A Network Model of the Mathematical Concept of Area

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The concept of area is important in mathematics learning, as it is the entry point to the deep connection between multiplicative and geometric structures, and requires integrating numerical and geometric understanding. The notion of area as an array of units is central to understanding the area concept. Previous works show that array representations can enhance: 1) learning the spatial structuring of units in the area concept, and 2) the understanding of the two-dimensional nature of the multiplicative process. Extending this dual (spatial and numerical) role played by arrays, we reformulate the area concept as a network, requiring a coagulation of spatial and numerical concepts. Starting from this theoretical model, we developed a spiral of physical manipulations, addressing different aspects of the area concept, to explore whether and how students tie together the different individual concepts in the network. Preliminary results show that students implicitly use multiplication concepts while solving an area problem, but the application of the individual concepts to the area concept is implicit, inconsistent and unstable. We argue that these results support our network model of the area concept.

Introduction

The problem of understanding the mathematical concept of area is usually seen as involving coordination between spatial and numerical representations (Sarama & Clements, 2009). At the operational level, many tasks seek to help the student transit from a qualitative (geometrical or spatial) to a quantitative (numerical) way of thinking about spaces, through the process of partitioning a space, and then arraying the unit parts (Battista, 2007). However, while measuring the area of a rectangle, which provides the foundation for the quantification of area, apart from a proper recognition of the unit and its relation to the given measure, one also needs to know the array structuring of units, and understand area as the product of the rectangle's length and breadth. Thus, the array concept forms the basis for both 1) area-measurement, and 2) the role of multiplicative thinking in calculating area.

In a different vein, many studies report the importance of the array concept for learning multiplication itself, and thus developing multiplicative thinking (Loveridge & Mills, 2009; Barmby, Harris, Higgin, & Sugget, 2009). This is wider than multiplication, and involves a broad range of topics, such as understanding the inverse relation between multiplication and division, part-whole relation, fractions, proportions, etc. It is interesting to note in this context that similar to area, some of these concepts involve spatial relations. Multiplicative thinking leads to a multiplicative response to a situation, by identifying or constructing the multiplicand, the multiplier and their simultaneous coordination in that situation (Jacob and Willis, 2003). Loveridge & Mills (2009) elaborate on Sophian (2007) to argue that the (conceptual) structure of multiplicative thinking has a proportional basis, which involves equal-sized parts or groups, while additive thinking has a part-whole structure, with unequal-sized parts. They further state that the concept of a unit plays an important role in the understanding of multiplicative relationships, which usually develops from the context of additive reasoning. According to this view, the need for multiplicative relation is realized when the units of quantification is other than one (i.e. either a composite unit or a fractional unit). Sophian (2007) claims that young children lack the understanding of constant units while doing equal sharing, and tend to divide a continuous quantity into some specific number of pieces, ignoring the size of the pieces (cited in Loveridge & Mills, 2009).

Davydov (1991) points out that multiplication, as an operation, is distinct from addition, and has measurement in practical contexts as its source (cited by Clark & Kamii, 1996). Barmby. et al. (2009) suggests that addition and subtraction could be viewed as unary operations, with each input representing the same quantity, while multiplication needs to be seen as a binary operation with two distinct input quantities. They state that the four important aspects of multiplication are: replication, binary nature of multiplication (as two different quantities), commutativity, and distributivity. They propose that array could be considered as a key representation for multiplication, as it helps in moving from an operational or process view, to a structural view of multiplication. Unlike array representation, equal-grouping and number-line representation encourage unary thinking and repeated addition method respectively, but miss two important aspects of multiplication, namely commutativity and distributivity. For example, when the numbers are swapped in these representations, they look quite different, and the commutative law is not immediately obvious. The array representation encourages students to think about multiplication as a binary operation, with rows and columns presenting the two inputs.

This short review suggests that the array structure can be used to facilitate the understanding of both area-measurement and multiplication, and in turn, that multiplication and array are key components of the area concept. Based on this interconnected relation between area, array and multiplication, we develop a view of the area concept that is different from the one suggested by the standard model of a learning trajectory where the learner moves from a qualitative to quantitative understanding of space. The interactions between the qualitative and quantitative concepts may be complex in this view, but the learner's progress is usually viewed as a linear trajectory. In contrast, we propose a network model of area, and a spiral model of learning this concept, where many different passes through the components of the network leads to an integration of the concepts, and thus a proper understanding of the concept of area.

Area: A Network Model

We view the (initial) area concept as a network concept, requiring the coming together, or coagulation, of four ideas – unit, array, multiplication, and unit of units. The last could be viewed as an array of arrays, and we introduce below a broader concept related to this, which we term extrapolation. Students usually do not have a fully developed understanding of these individual ideas when the area concept is introduced. Also, these concepts could be: partially or implicitly known, partially stable, understood in an intuitive/qualitative fashion, connect to each other in unstable ways, and some of them may not be known. Figure 1 presents a concept map of this network notion of area.

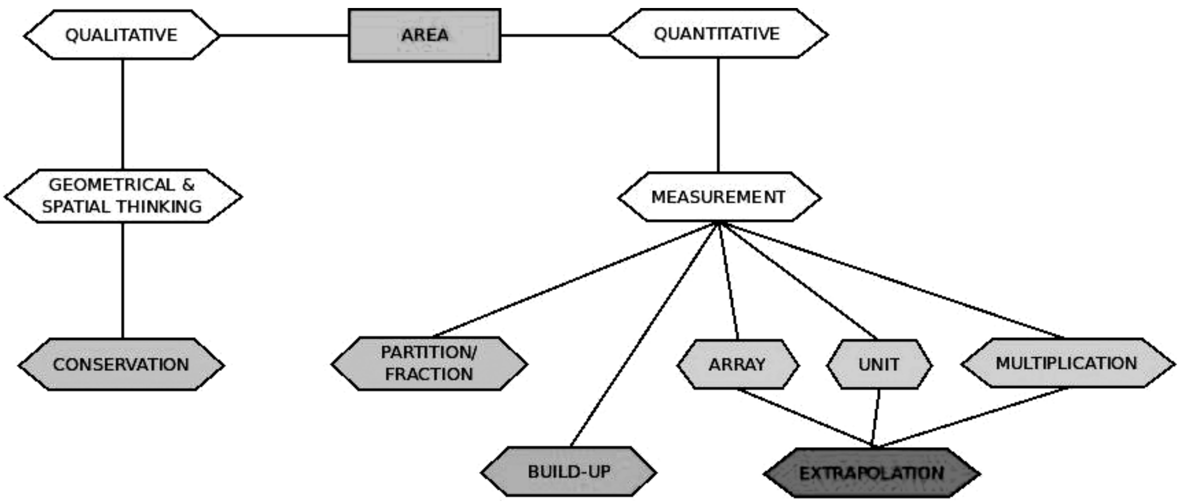

Figure 1. A network representation of the area concept.

The elements (Array, Unit, Multiplication) on the right indicate the standard concepts/operations associated with area. The elements (Conservation, Partition/Fraction) indicate tasks used to facilitate the understanding of area. The element (Extrapolation) indicates a new concept/operation we introduce. The element (Build-Up) indicates a new task we have developed to explore area.

Extrapolation

In relation to the network notion of area, we introduce a new concept/operation –extrapolation, which we consider a key part of understanding the area concept. Extrapolation refers to the use of the multiplication algorithm to calculate the area of rectangular spaces that are very large (or very small, folded etc.), where physical arraying is impossible or difficult. This operation, we believe, is one of the key objectives of learning to calculate area using the multiplication algorithm. Further, the ability to do this operation is an indication that the learner has consolidated her understanding of the connection between area, array and multiplication.

We would like to note here how this view is different from the standard way of testing the area concept, which usually involves calculating the area of complex figures made up of standard shapes, such as an L-shaped figure made of two rectangles. This task is a variant of the partitioning task, requiring the learner to imagine partitioning the given complex figure, and consider it as being made up of some shapes whose areas can be easily calculated, and then applying the multiplication algorithm to each partition (given some values for the sides), and then adding the results.

The extrapolation test, on the other hand, requires the learner to imagine how the area of a large space (such as the floor of a room) could be *measured* using a known shape (such as a square). The multiplication algorithm is then applied twice, first to her known unit, and then to the larger unit measured by it. This operation requires a deeper understanding of the array structure, where any space is seen as an array of units, and any given unit can be used to build up an array. Further, it requires understanding the relation between multiplication and arrays, as the unit is used to measure only the *borders* of the larger space. It also requires a deeper understanding of the relation between multiplication, array, and measurement, where the multiplication operation used to calculate the area of the given unit is extrapolated to a wider space through the use of the given unit as a measurement unit. Ideally, learners who understand the area concept should be able to move quickly to this operation. We believe that the extrapolation operation should be one of the key objectives of learning the area concept, and this operation needs to be supported by tasks designed to teach the area concept.

The present study explores how the above network of related concepts involved in area and multiplicative thinking could be checked, and also built up, through a task series based on physical manipulations.

Intervention Design

Physical manipulation of objects has been suggested as one way of facilitating the understanding of abstract mathematical concepts with spatial elements, such as fractions (Martin & Schwartz, 2005). The authors argue that physical actions can support symbolic learning, as learning of abstractions require reinterpretation, and this is difficult by just thinking. They have proposed four ways in which physical actions could support learning and thinking (Induction, offloading, re-purposing and physically-distributed learning). Physical actions play these supporting roles based on the degree of stability in ideas and environments. This relation is captured by figure 2.

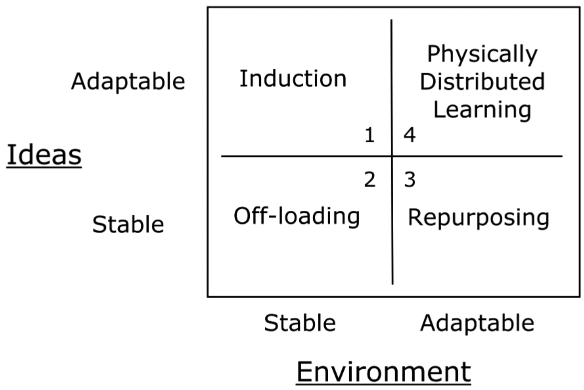

Figure 2. Drawn from Martin & Schwartz (2005)

Based on this conceptualization, the authors show how some physically distributed learning tasks leads to the learning of fraction concepts. This intervention, and the interactionist model of cognition underlying it, has inspired the design of our tasks reported in the next section. However, we consider the authors' way of characterizing the link between concepts and actions adequate only for developing interventions that address the fraction concept. Given our network view of the area concept, we consider this account insufficient as a starting point for designing physical manipulation tasks that facilitate the understanding of the area concept. This is because it is unclear what 'stable' means in relation to a network concept.

As we discussed above, the different concepts involved in the network may be partially known, stable to different degrees, or their interconnections could be unstable. To address this 'patchy knowledge' problem, we believe the physically distributed learning paradigm needs to be extended, to include a series of tasks, which, when done in an interconnected and spiral fashion, seek to bring closer together the ideas involved in the area concept. We outline below a pilot study that investigated such a task series, which we developed and explored with 5th grade participants.

Pilot study

To explore the network notion of the area concept, we developed four tasks, organized in a spiral, and roughly mapped to the four components involved in the area concept: unit, array, multiplication, and unit of units (extrapolation). One objective of this task series was to have a physical and observable process, so that we could keep track of how students progressed through each individual task, as well as the series. The methodology adopted for this study is microgenetic, since we were focusing on the processes involved when students were doing the tasks. Task-based interviews were done either in school or in the research institute. Interviews were video recorded with prior consent from the students and their parents. The video data was used for analysis.

Sample

The participants were grade five students, ten from one school and nine from another school. The tasks used were similar, but not identical, across all the students, as the tasks were progressively modified along the course of the study. The response and approach adopted by the initial students helped in the refinement of the tasks. This paper reports an analysis of eight students, four from each of the two schools, as these students were exposed to the exact identical tasks reported in the next section.

Tasks

1. Qualitative comparison task: This task required students to compare two pairs of rectangular sheets, with a small difference either in length or breadth, and find which sheet was bigger. The aims of this task were: (a) to prime the student with rectangular sheets, and (b) to explore whether the students were sensitive to the two-dimensional nature of area, or whether they only focused on one-dimension, such as length or breadth.

2. Build-up task: This task involved building up a rectangle using a set of squares. The number of squares was carefully chosen, to map to the multiplication table, or not. This task had a set of objectives. One, it sought to understand how students related numbers to rectangles. Two, it explored whether, and how, students used the multiplication table while making an array structure. Three, it sought to prime students about the relation between multiplication and arrays.

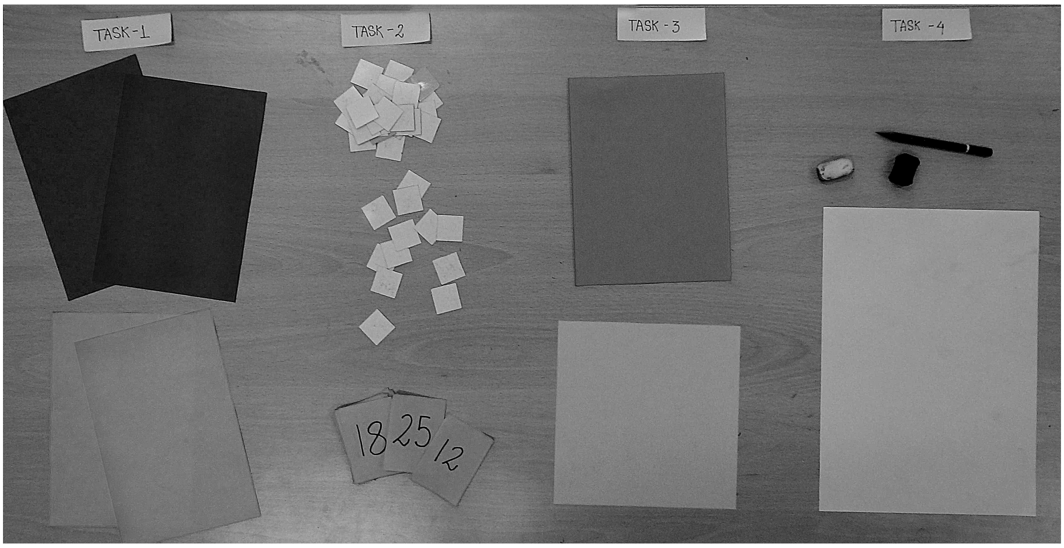

Figure 3. Materials used for the tasks

The task proceeded as follows for each student. The student was first shown small sheets with different numbers written on them. These were held with numbers facing the experimenter, and the student was asked to pick one sheet. Once the number was picked, the student was given that number of unit square cards (1inch×1inch), from a box of such cards. She then had to build-up a rectangle with those many cards. The student first physically arranged the cards to get the rectangle. She was then asked the number of cards in the length and breadth of the resulting rectangle.

This task is the opposite of the standard partition task, and seeks to build up a geometrical figure from a set of parts. The flexibility provided by physical manipulation potentially allowed students to develop rectangular array structures with various dimensions, and thus understand the relationship between the given number, the dimensions of the rectangle, and multiplication. Since the task requires overt action on the part of students, it also allowed us to infer their understanding of the multiplicative relation from their actions.

This task had a second phase, where, after a few trials, students were given a number, and before they physically created the rectangle by arranging the cards, they gave verbal responses about the possible rectangle.

Particularly, whether they could make a rectangle with the given number of cards, and the number of cards that should be used along the length and breadth.

In this second variation, students were shown one of the following multiples or primes (10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 11, 13, 17, 19, 23, 29) in random order, but starting with the simple composite numbers.

The aim of this variation was to prompt them to see how the given number could or could not decompose into factors yielding the array structure, by asking them the number of cards in the length and breadth. This could prime the multiplicative relation between the given number (of cards), and the way its factors correspond to the length and breadth of the rectangle.

3. *Quantitative comparison task:* This task returns to the comparison operation (thus creating a spiral in the task series), but now the comparison is more complex, and informed by the numerical and spatial manipulations done in the build-up task. The task was to compare a given square sheet (7inch×7inch) with a given rectangular sheet (8inch×6inch) and find which is bigger. Students were prompted with a context, e.g. "I need to cut squares of this shape from the rectangle" to help them comprehend the task. Since the shapes are different, qualitative comparison is difficult (e.g. overlapping). To build on the previous task, the students were given a square card (1inch×1inch) and were asked to use it. After the build-up task, this measuring task allows us to explore the various strategies (e.g. array structuring, complete covering, multiplication, etc.) used by students while measuring the sheets.

The aim of the task was to see whether the previous build-up task (array structuring) helped them in understanding the relation between area and arrays, to the point where they could think of comparison in a numerical fashion. The task also allows seeing whether the students use the multiplicative or addition relation to get the measure of the two areas.

4. *Unit of units (Extrapolation) task*: This task tested whether students could do the extrapolation operation. It represents another spiral in the series, as it creates a version of the build-up task, using the elements from the previous comparison task. The task required using the rectangular sheet of the earlier task, to get an area measure of an A4-sheet, and then using the A4-sheet to get the area of a table. The students were given an A4-sheet, and they were free to use the materials used in the previous task.

This task sought to explore whether students could extrapolate their understanding of area-measurement to bigger rectangles , and use an efficient unit for measuring. This task also creates the need to optimize the number of operations, as it is difficult to measure the table using the small square unit. This means the students have to think of the nested multiplicative relation.

Preliminary Results

We report findings from a detailed analysis of the responses of eight students, four from each school. A few events from earlier interviews (of the remaining students) are also presented.

1. In the qualitative comparison task, there were minute differences either in length or breadth between the sheets in each pair, which one cannot make out by merely looking. Students were shown the two sheets and were asked to find out which covered more space on the table. Initially, except for one, all the eight students compared either length or breadth of the rectangular sheets. Later they overlapped the sheets to compare them.

Students initially try to gauge area using one dimension, either the length or the breadth of the rectangle, rather than exploring area in two dimensions right away. This indicates that there are two different levels, possibly based on processing load, to the spatial understanding of area. There is a surface, 'first-pass', level that is based on one dimension. The learner moves to two dimensions only when this evaluation fails.

2. For the build-up task, all students did arraying (arranging either in row or column) to get the rectangular structure, for most numbers. Of the eight students, four found that the number of cards in the length and breadth could be found by multiplicatively splitting (factorizing) the given number. These four students explicitly used the multiplication table for the card task, while the other four were unsure about the multiplication tables. For the other four students, the use of multiplication was more implicit. We infer from their actions during the task that they were also using the multiplicative splitting for several numbers. For instance, a student started placing 5 cards in a row when given 35 cards, and made 7 such rows to have a 5×7 rectangle. But when asked whether the rectangle could be started with 6 instead, she said the arrangement will be one card short, without physically checking. She was not able to explain why she started with 5 in the first case, and how she knew about the second situation with 6 cards. But her actions, and the process used to arrive at the result, allows us to infer the use of multiplication table in making the array structure. In some instances, students moved between multiplication and addition while making the array. For instance, one student made 4×3 and 6×2 rectangles

physically with 12 cards, and said 15 cards can make a rectangle (prior to using the physical cards) as "3-5 za 15". But the same student, when given 10 and 13 cards later, said 3, 7 and 3, 10 are the sides of the rectangle respectively. Another student who said "7-4 za 28" for the number 28, also said 8 squares in length and 6 squares in breadth for the same number of cards, and wrote 8+6=14 and 14×6=64 on paper.

In some cases, students made only the boundary of the rectangle, by arranging the cards along the perimeter of a rectangle, leaving a gap with no cards inside. They were then asked to fill the rectangle. This move seems to suggest that there are two spatial understandings of a rectangle, one with an empty space in the middle, and the other as a fully filled space.

The task also allowed some students to explore rectangles with fractional lengths. For instance, two students cut the cards into equal parts to get rectangles, of which one made a 7.5×2 rectangle from 15 cards by cutting one card into half. The other one suggested a 5¼×4 rectangle for 21 cards and made a 6½×2 rectangle with 13 cards.

Overall, the task showed that students' understanding of the concepts involved in area were implicit, partial and unstable, particularly that of the connection between multiplication and arrays.

3. For the quantitative comparison task, students were provided with a unit card and were asked to compare a rectangle (8 inch x 6 inch) and a square (7 inch x 7 inch). Most students did not use the unit card initially, and were unsure about what to do next. They were either comparing the sheets qualitatively, or asked for a ruler to measure them. But when they were prompted to find the number of cards that can be made out of the rectangular sheet, all the students used the cards to make marks on the adjacent sides of the rectangle. Five students made marks along one of the dimension (length or breadth) and then added that number repetitively along the marks made on the other dimension of the rectangle. The other three students multiplied the number of cards that can fit on the adjacent sides of the rectangle to get the total number of cards. In the initial interviews, most students used the addition method, rather than the multiplicative method.

One student started with a qualitative comparison of two sheets by overlapping the sheets. When he was asked to explain, he compared the extra space left on the sheets after overlapping the sheets, and said that they are equal. He noticed that the width of the space left is one unit in each case. However, the student failed to notice that the extra space of the square can hold 7 unit cards, while the extra space of the rectangle can hold only 6 unit cards.

Overall, these results indicate that the multiplicative relation (factor-based split) used to create arrays in the build-up task was not used much in this task. Also, since they thought of using the unit card only after prompting, it appears that the creation of the arrays in the build-up task did not lead to thinking of a given space as an array.

4. For the extrapolation (or, unit of units) task, students had to extend the idea of area to a larger space. The task explored how students used units to measure the given space. Six students were able to do this task. All of them used the bigger unit to measure a given space. Three of them used the multiplicative relation, while the other three used the additive relation. For instance, one student found that 100 unit-square cards filled an A4 sheet, and a table can be filled with ten A4 sheets. She derived the number of cards for the table by adding 100 ten times, rather than multiplying 10 with 100.

Some students made errors such as placing the A4 sheet lengthwise along both the sides of the table, to get the number of such A4 sheets that could fit into the table. They then got the number of A4 sheets required to cover the table, either by just counting the number of such sheets along the boundary, leaving the inner space, or multiplying the number of sheets along the two dimensions. But the students were not consistent in this strategy, and changed their strategy when asked to explain how they got the total number of (the given square) cards in the table.

When students were asked how many times bigger the table was compared to the A4 sheet, most students were not able to comprehend the question. When the question was reformulated, to ask how many such sheets could fill the table, or how many small unit-square cards could be cut out of a rectangular sheet, they attempted the problem. The 'how many times?' question explicitly looks for the multiplicative understanding of a situation, while the revised framing may or may not be interpreted in the multiplicative term. Multiplicative understanding facilitates the idea of extrapolation, but it was challenging to explore this idea among students.

Overall, these observations seem to suggest that students do extrapolate the array structure to larger spaces, but they find the extrapolation of the multiplication operation difficult, and use the additive operation instead. Secondly, instances like putting a rectangular unit lengthwise along the table suggests that those students do not understand the geometrical mapping required to do this task correctly. This mapping is not a problem when using the square units given in the build-up task. This, in turn, suggests that students' extrapolation of the array structure is not a general one, but is influenced by square units.

Discussion

The most interesting pattern emerging from the results is the various levels of stability in learners' ideas (from task 2) and the inconsistency in the application of ideas learned in the previous tasks (tasks 3 and 4). This pattern of unstable and partial knowledge is better accounted for by our network model of the area concept (where understanding the concept of area requires a coagulation of four different concepts), than accounts that treat the understanding of the area concept as involving a (linear) shift from a qualitative to a quantitative notion of space.

The sequencing of the tasks in a spiral fashion did not entirely achieve the objective of interconnecting different aspects of the area concept, but it helped in revealing some of the issues involved in coagulating the individual concepts involved in the understanding of area.

Finally, from a methodological perspective, the physical and manipulative nature of the tasks provided us with a process understanding of students' thinking and learning involved in the area concept, particularly the use of multiplication in creating the array structure.

Based on Martin & Schwartz (2005), the physical manipulation tasks probably helped the students in understanding the array structure better, but we currently do not have evidence to support this view. More studies and analysis needs to be done to understand this process better.

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