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EXPLORING THE CONNECTION BETWEEN MULTIPLICATIVE THINKING AND THE MEASUREMENT OF AREA

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The understanding of the area concept requires connecting multiplication to geometry. Multiplicative thinking is a well researched area in mathematics education and has application in a broad range of topics. In the present study, we explore different ways in which multiplicative thinking is involved in the geometric measurement of area. Specifically, the focus is on developing tasks that elicit the use of multiplicative thinking in finding the area of geometric figures. We report tasks developed for two pilot studies along with student responses, where we explore the connections between numerical and geometrical aspects of area-measurement using multiplicative thinking.

Keywords: Multiplicative thinking, geometric measurement, area measurement, unit of units, array structure.

INTRODUCTION

Multiplicative thinking is a well researched area in mathematics education. Jacob and Willis' (2003) analysis suggests that multiplicative thinking leads to a multiplicative response to a situation by identifying or constructing the multiplicand, the multiplier and their simultaneous coordination in that situation. It involves attending to the multiplicative relation between quantities and magnitudes, and the capacity to mathematically deal with such situations (Subramaniam, 2011). Multiplicative thinking has application in a broad range of topics like understanding the inverse relation between multiplication and division, part-whole relation, fractions, proportion, etc. In contrast, the domain of measurement is relatively less researched, with even fewer studies that explicitly discuss the connection between measurement and multiplicative thinking.

Geometric measurement involves deriving a new quantity "the number of units", from the known quantities – magnitude of the unit and magnitude of the space to be measured, between which there is a multiplicative relation, namely, that the target magnitude is "so many times" the unit. Thus unlike in the case of direct counting of discrete quantities, multiplicative thinking lies at the heart of the concept of measurement. Lamon (2007) and several others have argued that the way measurement is handled in the elementary curriculum leads students to do an act of measuring rather than develop the *concepts* of measurement. She reports that very few students could understand that the unit of measure could be broken further into smaller subunits to make the measurement more precise. In the present study, we explore different ways in which multiplicative thinking is involved in the geometric measurement of area, with a specific focus on developing tasks that relate the use of multiplicative thinking in finding the area of geometric figures.

In the literature on multiplicative thinking, most of the situations and contexts examine proportionality and involve a linear relation between two single dimensional measures (for example, the relation between cost and weight, time and wage, speed and distance, etc.). Each such single dimensional measure is analogous to length; so many of the concepts explored in the proportionality context have their analogues in the case of the geometric measurement of length. For example, unitization, the process of mentally chunking discrete units into either a larger convenient unit (chunked unit), or breaking a unit into smaller units, plays an important role in proportional reasoning (Lamon, 2007). Unitization is also the basis of measurement, and flexible unitization is involved in tasks that require construction of a “unit of units” (Reynolds & Wheatley, 1996). The number line, which is a direct representation of length, is useful in reasoning in proportionality contexts. The double number line in particular is a convenient representation of proportional relationships (Subramaniam, 2008), which affords the structuring and co-ordination of subunits and chunked units. Battista (2007) has suggested that working with fractional units may help children understand the principle of unit structuring and unit iteration in measurement, which is similar to the process of unitizing.

The five measurement principles stated by Curry, Mitchelmore, & Outhred (2006) are: need of congruent units, use of an appropriate unit, using the same unit for comparing objects, relation between the unit and the measure, and structuring of unit iteration. Each of the above five principles requires appreciating the multiplicative relations that arise in the context of geometric measurement in various ways.

Progressing from the case of linear measures to other kinds of geometric measurement, we find that multiplicative thinking is involved in further ways. In the case of area measurement, multiplicative thinking arises firstly, in ways similar to length measurement: (i) the use of sub-units and chunked units (unit of units) in determining area (ii) inverse relation between size of unit and the measure. It also arises in ways that do not occur in the case of length measurement such as the array structuring of units in the case of rectangles leading to area as the product of length and breadth. Further there is a multiplicative relation between the area of the rectangle and the unit, between the area and length, and between the area and breadth. Correspondingly, there is an inverse relation between the area measure and the magnitude of the area unit, which is itself dependent on the length and breadth of the unit. Further, the passage to non-rectangular polygons involves triangulation starting from the area of a right triangle obtained by dividing a rectangle in half, which involves a multiplicative relation. Thus we find that multiplicative relationships are involved in complex ways in area measurement.1

1 . We hypothesize that some of this complexity carries over into situations involving multiple proportions (e.g. food required is proportional to number of people and number of days). However, an investigation of this question is beyond the scope of our study.

GEOMETRIC MEASUREMENT: AREA

The present study tries to explore the need of multiplicative thinking in understanding and connecting the spatial attribute (e.g. area) and the numerical value assigned to the spatial attribute, through an analysis of students' strategies in solving area tasks. One of the goals of the study is to develop tasks that elicit different ways in which multiplicative thinking arises in area measurement. An example of such a task is found in Reynolds & Wheatley (1996), where a fourth grader solved the problem of finding the number of 3-by-5 cards required to cover a 15-by-30 rectangle by dividing the area of the rectangle by the area of the card. The researchers had initially missed the point that the student had actually realized that the number obtained after dividing the areas would be correct only if the two dimensions (length and breadth) of the small card completely divide the two dimensions of the large rectangle respectively (cited in Battista, 2007). This gives an instance where the unit is related to the target measure of area of the space not only in terms of the multiplicative relation between the magnitude of the unit and the magnitude of the target measure, but also in terms of the multiplicative relation between either the length or the breadth of the unit to the target measure (i.e., area of the space to be measured).

PILOT STUDY 1

We intend to report an analysis of students' responses in two pilot studies: pilot study 1 and pilot study 2. The study is done through task-based interviews of students. The goal of the two pilot studies was to develop tasks that explore multiplicative thinking in the context of area related tasks, and to collect preliminary data in the form of student responses.

The two pilot studies were carried out with different groups of students.

We present a description below of the three tasks that were used in pilot study 1. These were basically tiling tasks which require students to find the number of units which cover a given area. Task based interviews with students were audio recorded with their consent and were used for analysis. Pilot study 1 is based on the idea that unit structuring in the tiling task not only involves the numerical relation between the unit and the measure, but also the spatial structuring of the units.

The sample for pilot study 1 consisted of a mixed ability group of ten Grade 5 students from two schools in the neighborhood – five from each of the two schools. The students were identified by their respective class teacher as of above average ability (5 students), average ability (3 students) and below average ability (2 students).

Tasks

In pilot study 1, we developed and presented three versions of the tiling task (Table 1) similar to the one used by Reynolds & Wheatley (1996). The dimensions of the tiles were chosen so that although the area of the tile is a factor of the area of the rectangle to be covered, each dimension of the tile is not necessarily a factor of the dimensions of rectangle (see Table 1). For example, in the first task with the second card (3cm□2cm) the breadth of the tile is not a factor of the length (see Figure 1).

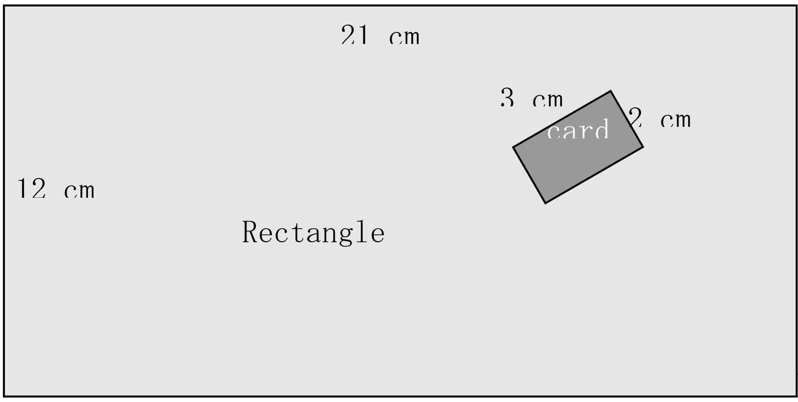

Figure 1. Tile's breadth (2 cm) is not a factor of rectangle's length (21 cm)

For each of the tiling tasks, students were asked whether a given tile, pasted repeatedly, could cover the rectangle and the number of tiles required. For the first task, students were given physical objects – a rectangular sheet and three different paper tiles, one by one. For the other two tasks, students were only verbally told the dimensions of rectangle and tile. The last task had a right triangular tile and students were also asked to find other shapes that could be used for covering the rectangle.

Table 1: Pilot test questions

Task	Given	Dimension of the rectangle	Shape and Dimension of the Tile(s)
1	Rectangular paper sheet and three different paper tiles	21cm□12cm	Rectangle, 2cm□2cm, 3cm□4cm, and 6cm□2cm
2	Dimension of the Rectangle and the Tile	19m□6m	Rectangle, 3m□2m
3	Dimension of the Rectangle and the Tile	15m□8m	Right angled triangle, height (5m), base (2m)

Analysis of the pilot study 1

Two kinds of strategies were observed when students solved the first two tasks.

Dividing the area (found by multiplying the length and breadth) of the rectangle by the area of the tile, to know whether the tile (without breaking) could be used for covering the rectangle, and then getting the number of such tiles. Four students used this strategy to solve the tasks. These students did not make a correct judgement in the cases where the dimension of the unit is not a factor of the dimension of the rectangle. They may not have grasped the spatial structuring of the tile in such circumstances.

Checking the tile along the two dimensions of the rectangular sheet. Students using this strategy were correct in tasks based on the fitting of the rectangular tiles. Four students used this strategy in the tiling task, but not all of them were consistent in using this strategy throughout the tasks.

Two students initially started with the second strategy, but ended up using the first strategy for the remaining problems. They are counted among the students using the first strategy. These instances clearly show that it is not enough to know the multiplicative relation between the area of the tile and the area of the rectangle but also the multiplicative relation between dimension of the tile and dimension of the rectangular surface. For the last task, every student said that the triangular tile cannot completely cover the rectangular space in the first instance, with two students changing their response later, realising that two such triangles can be joined to obtain a rectangle. The students could be said to be aware at least implicitly of the multiplicative relation arising from the geometric division of a figure into equal parts. The pilot study highlights the instances where one needs to have the ability to connect the multiplicative relation between the dimensions with the spatial structuring of the tile to get a sense of why the procedure works in some cases and not in other cases.

Findings of Pilot Study 1

Eight students knew that the area of the rectangle is the product of its length and breadth and used this relation to obtain the area. Further they could relate the covering problem to the area since they divided the area of the rectangular sheet by the area of the unit. However, they did not respond correctly to all versions of the task because they failed to check if the dimensions of the unit divided the dimensions of the rectangle separately. This requires the students to *visualize* the covering as a structured array. Given this, it is not clear if the students who used the formula for the area of a rectangle even connected this with the array of unit squares covering the rectangle. It is also not clear whether the physical action of tiling with a given unit helped students to arrive at the right response to the tiling tasks. Hence in pilot study 2 we developed a set of tasks that examined more centrally the array of square units.

PILOT STUDY 2

Pilot study 1 suggested the need for further exploration of the connection between multiplicative thinking and the measurement of area and hence the need was felt to develop tasks that can elicit such forms of thinking. In pilot study 2, video data of task-based interviews of students were collected with consent from the student and their parents. Interviews were done either in school or in the research institute. Video recordings of the interviews were used for analysis.

Sample

The sample consisted of 10 students studying in Grade 5 in a school serving mostly middle income families and 9 students of Grade 5 in another school serving mostly low to middle income families. Although the tasks used were not identical across all these students since they were progressively adapted in the course of pilot study 2, the approaches taken by the students to do the tasks were broadly comparable. Four students each from the two schools received exactly similar tasks as mentioned in the next section.

Tasks

There were four tasks, which are explained briefly below.

Comparison Task: This task required students to compare two pairs of rectangular sheets with very small differences either in length or breadth but not both. The aim of this task was to prime students with rectangular sheets and to explore whether the students compare area by overlap or by comparing only the attributes like length or breadth.

Card Task: Constructing a rectangle with a given number of unit square cards (1inch□1inch). In this task students were first shown a number and were then allowed to take those many square cards from a box to make a rectangle. This task provides the possibility to connect the number of cards and the resulting rectangular array, and also seeks to exploit the flexibility of physical manipulation. This task implicitly requires one to notice the multiplicative relation between the given number and its factors along the length and breadth. Since the task requires overt action on the part of students, it allows one to see whether students are implicitly attending to the multiplicative relation involved, even if they do not overtly express this relation.

Measuring Task: Comparing two sheets to decide which is larger – a square sheet (7inch□7inch) and a rectangular sheet (8inch□6inch). The difference in area between these sheets is small, and cannot be determined directly by overlap. Students were also given a small square card (1inch□1inch) and asked to use it if they needed to. After the card task, the measuring task allows us to explore the various strategies (e.g. array structuring, complete covering, etc.) used by students while measuring the sheets. Further, this task allows us to look into whether the students apply the ideas abstracted from the previous tasks, i.e., whether they use the multiplicative relation or repetitive addition to get the measure of the two areas.

Unit of units Task: Using the rectangular sheet of the earlier task to get the measure of an A4-sheet, and then using the A4-sheet to get the area of a table. This task was used to explore whether the students could extend their understanding of area-measurement to bigger shapes. Further, this task may create the need to optimize the number of operations and thus use the nested multiplicative relation.

Procedure

The tasks were presented verbally together with the material. At times students were prompted with a context (e.g. “I need to cut squares of this shape from the rectangle”) to engage them during the interaction. For the comparison task students were given two pairs of rectangular sheets. For the card task they were shown a specific number written on small sheets. For the first set of trials, students were shown one of the following composite numbers: 10, 12, 14, 15, 16, 18, 20, 21, 24, 25. They were then asked to take those many cards from a given collection of cards to make a rectangle. In the next set of trials, the student were given a composite number and were asked to respond verbally about the rectangle that could be made from the cards. Finally students were shown either a prime (11, 13, 17, 19, 23 or 29) or a composite number, and were asked whether they could make a rectangle with the given number and how many cards would be there along its length and breadth.

For the measuring task, students were given a square sheet, a rectangular sheet, and a unit square card. For the unit of unit tasks students were given an A4-sheet and were free to use the materials used in the previous task. The use of actual physical or concrete materials provide students the flexibility to work with the given objects in the way they want. In the case of the card task they could rearrange the cards to explore various possible rectangles and squares that can be constructed, thus allowing students to make various arrangements and explore connections between numbers and the rectangle figures.

Findings and discussion

We report findings from a detailed analysis of the responses of eight students who were presented with stable versions of the tasks. A few instances of the earlier interviews of the remaining students will be presented to give a picture of how the ideas for the tasks emerged during the study.

In the first comparison task, there were only minute differences in either length or breadth between a pair of sheets, and these could not be determined by merely looking at them. Among the eight students, all except one tended to compare the rectangular sheets either by length or breadth, when the sheets were placed flat on the table next to each other. However later they overlapped the sheets to compare them. This suggests a natural tendency to compare the sides of rectangles when asked to compare the sizes, and may indicate an implicit understanding of the relation between the sides and area.

For the second (card) task, four students (after a few trials) understood the connection between the factors of a given number and the resulting rectangular shape. For the other four students this connection was either implicit or unstable. It appeared that in some of the trials, they were implicitly using the multiplicative relation between the number of cards and the resulting arrangement. For example, in several instances students created the first row of cards using a number that was a factor of the given number. However, they were not able to explain why they chose that number. The connection was unstable for some students, who were not consistent with their strategy. For instance, one student made 4□3 and 6□2 rectangles with 12 cards and said 15 can be made into a rectangle as "*3-5 za 15"* (i.e., 3 times 5 is 15). Later, when asked about the sides of the rectangle that can be made with 10 and 13 cards respectively, he said 3, 7 and 3, 10. Another student who said "*7-4 za 28"* for the number 28, also said 8 squares in length and 6 squares in breadth for the same number and wrote 8+6=14 and 14□6=64 on paper. This showed that students shift between the additive and multiplicative relations between numbers while doing this task.

The card task also allowed some students to explore rectangles with fractional lengths. Two of the eight students cut the cards into half to get rectangles: one student made a 7.5□2 rectangle from 15 cards by cutting one card into half. Another student suggested a $5\frac{1}{4}$ □4 rectangle with 21 cards and made a $6\frac{1}{2}$□2 rectangle with 13 cards.

In the card task, one student among the eight students said that he is looking at the factors of the number for making the rectangle. Three other students explicitly used the multiplication table for the card task. The remaining four students, although not able to express their idea or thought process, indicated through their actions the implicit use of multiplicative relation between the given number and its factor. These students started with a factor of the given number, but were not able to say why they started with those number of cards. In fact, one student said the number came to her by own without thinking. In another instance, the student said square can be made with numbers which will come *double* e.g. *6-6 za, 1-1 za, 10-10 za, 3-3 za.* This showed that the student had an idea of how a number is related to a square arrangement. Students with more awareness of the multiplication facts did get the measurement-multiplication connection sooner than those without or those relying more on addition facts. We infer this because all the four students who decomposed the number into factors were fluent with the multiplication tables, while the other four were unsure about multiplication tables.

In some instances when the students were not able to find the factors for a number, they tried to arrange the cards along the perimeter of a rectangle leaving a gap in the centre. In such cases, students were asked to make a complete rectangle that is fully covered. However this move indicates that the task does not necessarily constrain students to making a figure by filling a rectangular space with cards. The perimeter arrangement is interesting because students then decompose a given number using both additive and multiplicative relations as seen earlier in the instance where a student decomposed 28 as 2□(8+6). This also indicates that a rectangle is imagined in two ways, one as an array (or filled space) and the other as a border (with empty space). An interesting question is how these two ways of conceptualising the geometric figure influences the learning of the area concept.

For the measurement task one student compared the extra space that was left on both the square and the rectangle once they were overlapped or placed one above the other, and saw that the width of the space left was one unit in each case. So the student said both the sheets have equal space. But the student missed the fact that the extra space of the square can hold 7 cards, but the extra space in the rectangle can hold only 6 cards.

All the eight students marked the adjacent sides of the rectangle using the given square card when they were asked to find the number of cards that can be made out of the rectangular sheet. Only three students multiplied the number of cards that can fit along the adjacent sides of the rectangle to get the total number of cards. The other five did repetitive addition to get the total number of cards. This was found even with the students interviewed earlier (in pilot study 2). Thus it appears that the most common method was to add the number of cards in one row repeatedly as they counted the card marks along the adjacent side.

Six students were able to do the unit of units task but in this case also three used the multiplicative relation while the other three used the repetitive addition relation. For example, once the student knew that an A4-sheet can have 100 cards, and a table can be filled with 10 such A4-sheets, then the number of cards for the table was arrived at by adding 100 ten times rather than multiplying 10 with 100 to get ten 100 cards.

Some students initially tend to find out the number of times a rectangle placed lengthwise cover the length and breadth of the table respectively. They orient the rectangle lengthwise even when they place the rectangle along the breadth of the table. The students then multiply the numbers they get, to obtain a wrong result for the number of rectangles that can cover the table. But the students were not consistent in this strategy and changed their strategy when asked to explain how they got the total number of (the given square) cards in the table.

It is worth noting here that for the unit of units task, the unit was not a standard square unit but some multiplicative (or chunked) unit. In such instances it is not enough to see the multiplicative relation in measurement, but one also needs to see the geometrical division of the measure in terms of this new multiplicative unit. In other words, the students need to coordinate both the numerical and geometric aspects to perform this task. One way to interpret the above strategy (of measuring lengthwise) is to consider the students as having an implicit understanding of the need for coordinating the two aspects, but not understanding the nature of the array structure. Perhaps the numerical aspect dominates, and the measurement is done to get values for multiplication, while a proper understanding requires keeping the multiplication and geometric structure in mind simultaneously.

CONCLUSIONS

The tasks used in the present pilot study provides students the possibility of directly connecting the measurement unit with the number. The five important insights from the present study are:

- Students most often were inclined to focus on the sides of the rectangle rather than space covered by it or the area of it. Thus in the comparison task students tend to compare the sides. Even for the card task, they often missed to fill the inside of the rectangle and place the cards either along the length, breadth or the boundary.
- Students often used the additive relation between the numbers in the card task, rather than the multiplicative one while splitting up the given number for constructing a rectangle.
- Even when students use the connection between multiplication and array structure in the card task, this strategy is not stable.
- Students have an implicit understanding of the link between numerical properties and area but they are unable to express this understanding. The tasks used in the present study allows children to manipulate many structures, giving us insights into their implicit thinking.
- From the method perspective, the tasks used in the present study gives students flexibility to explore various structures and their connections to numbers, and allow us to explore students understanding about multiplicative (chunked) units, even when they are unable to articulate their understanding.

In this report, we have analyzed the four tasks separately, and have not examined how the exposure to one task (say the card task) improves/impairs performance in the next task. We expect to report some of this more complex analysis in our presentation. The application of multiplicative thinking is already being explored in many other area of learning. The present study suggests that the understanding of measurement can be enriched by building its connection with multiplicative thinking. The tasks used in the study allows children to manipulate and explore many structures and their connections with numbers, and helps researchers examine systematically the relations between the different operations and components that together make up the area concept.

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