The Arithmetic-Algebra Connection: A Historical-Pedagogical Perspective

K. Subramaniam and Rakhi Banerjee

Abstract The problem of designing a teaching learning approach to symbolic algebra in the middle school that uses students' knowledge of arithmetic as a starting point has not been adequately addressed in the recent revisions of the mathematics curriculum in India. India has a long historical tradition of mathematics with strong achievements in arithmetic and algebra. We review an explicit discussion of the relation between arithmetic and algebra in a historical text from the twelfth century, emphasizing that algebra is more a matter of insight and understanding than of using symbols. Algebra is seen as foundational to arithmetic rather than as a generalization of arithmetic. We draw implications from these remarks and present a framework that illuminates the arithmetic-algebra connection from a teaching-learning point of view. Finally, we offer brief sketches of an instructional approach developed through a design experiment with students of grade 6 that is informed by this framework, and discuss some student responses.

Introduction

Mathematics is widely believed in India to be the most difficult subject in the curriculum and is the major reason for failure to complete the school year in secondary school (National Centre for Educational Research and Training 2006). The education minister of a western Indian state recently complained that students spend vast amounts of time studying mathematics, with limited success and at the cost

K. Subramaniam (✉)
Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai, India
e-mail: subra@hbcse.tifr.res.in

R. Banerjee
School of Social Science, Tata Institute of Social Sciences, Mumbai, India
e-mail: rakhi@tiss.edu

J. Cai, E. Knuth (eds.), *Early Algebraization*, Advances in Mathematics Education,
DOI 10.1007/978-3-642-17735-4_6,

of neglecting other subjects and extracurricular activities. Similar complaints pressured the state government into removing the mandatory pass requirement in mathematics for the school exit examination in the year 2010. This is reflective of a trend among some school systems in India to make mathematics an optional subject in the school exit examination. Students' difficulties in mathematics may however have deeper causes located in the education system as a whole, which need to be addressed on multiple fronts. The nation wide annual ASER surveys, based on representative samples of rural schools, found very low levels of learning of mathematics in the primary grades (ASER report 2010). A survey of the most preferred, "top" schools in leading Indian metro cities found surprisingly low levels of conceptual understanding in science and mathematics (Educational Initiatives and Wipro 2006).

Efforts to address the issue of failure and low learning levels include an important reform of the school curriculum following the 2005 National Curriculum Framework (NCF 2005), which emphasized child-centered learning (National Centre for Educational Research and Training 2005). New textbooks for grades 1–12 following the NCF 2005 were brought out by the National Council of Educational Research and Training (NCERT) through a collaborative process involving educators and teachers. We shall refer to these as the "NCERT textbooks". The NCERT textbooks in mathematics have introduced significant changes in the instructional approach, especially in the primary grades. However, one of the issues that remain inadequately addressed in the new textbooks is the introduction to symbolic algebra in the middle grades, which follows a largely traditional approach focused on symbol manipulation. Since algebra is an important part of the secondary curriculum, bringing mathematics to wider sections of the student population, entails that more thought be given to how algebra can be introduced in a manner that uses students' prior knowledge. Our aim in this chapter is to articulate a framework that addresses the issue of transition from arithmetic to symbolic algebra, and to outline an instructional approach based on this framework that was developed by the research group at the Homi Bhabha Centre through a design experiment. In this section of the paper, we shall briefly sketch the background of the reform efforts, insofar as they are relevant to the teaching and learning of algebra.

In India, school education includes the following levels of schooling: primary: grades 1–5, middle or upper primary: grades 6–8, secondary: grades 9–10 and higher secondary: grades 11–12. The provision of school education is largely the domain of the state government, subject to broad regulations laid down by the central government. The vast majority of students learn from textbooks published and prescribed by the state or the central government. Following the reform process initiated by the central government through NCF 2005, many state governments have revised or are in the process of revising their own curricula and textbooks to align them with the new curriculum framework. In comparison to the earlier years, the mathematics curriculum and the NCERT textbooks at the primary level have changed significantly, while the middle school curriculum, where algebra is introduced continues largely unchanged (Tripathi 2007).

Algebra, as a separate topic, forms a large chunk of the middle and secondary school syllabus in mathematics and also underlies other topics such as geometry or trigonometry. Students' facility with algebra is hence an important determinant of success in school mathematics. Thus, as elsewhere, algebra is a gateway to higher learning for some pupils and a barrier for others. In the new NCERT textbooks, formal algebra begins in grade 6 (age 11+) with integer operations, the introduction of variables in the context of generalization, and the solution of simple equations in one unknown. Over the five years until they complete grade 10, pupils learn about integers, rational numbers and real numbers, algebraic expressions and identities, exponents, polynomials and their factorization, coordinate geometry, linear equations in one and two variables, quadratic equations, and arithmetic progressions.

The grade 6 NCERT mathematics textbook introduces algebra as a branch of mathematics whose main feature is the use of letters "to write formulas and rules in a general way" (Mathematics: Text book for class VI 2006, p. 221). It then provides a gentle introduction to the use of letters as variables, and shows how expressions containing variables can be used to represent formulas, rules for a growing pattern, relations between quantities, general properties of number operations, and equations. However, this easy-paced approach gives way to a traditional approach to the manipulation of algebraic expressions in grade 7, based on the addition and subtraction of like terms. The approach in the higher grades is largely formal, with real life applications appearing as word problems in the exercises. Thus, although an effort has been made in the new middle school textbooks to simplify the language, the approach is not significantly different from the earlier approach and does not take into account the large body of literature published internationally on the difficulty students face in making the transition from arithmetic to algebra and the preparation needed for it. (For details and examples, see Banerjee 2008b.)

The NCERT mathematics textbooks for the primary grades, have attempted to integrate strands of algebraic thinking. In a study of the primary mathematics curricula in five countries, Cai et al. (2005), have applied a framework that identifies the algebra strand in terms of algebra relevant goals, algebraic ideas and algebraic processes. Some of the elements identified by Cai et al. are also found in the NCERT primary mathematics textbooks. There is a consistent emphasis on identifying, extending, and describing patterns through all the primary grades from 1 to 5. "Patterns" have been identified as a separate strand in the primary mathematics syllabus, and separate chapters appear in the textbooks for all the primary grades with the title "Patterns." Children work on repeating as well as growing patterns in grade 2 and grade 3. Many other kinds of patterns involving numbers appear in these books: addition patterns in a 3×3 cell on a calendar, magic squares, etc. A variety of number puzzles are also presented at appropriate grade levels; some of the puzzles are drawn from traditional or folk sources (for an example, see Math-magic: Book 3 2006, pp. 92–94).

Simple equations with the unknown represented as an empty box or a blank appear in grade 2 and later. The inverse relation between addition and subtraction is highlighted by relating corresponding number sentences and is also used in checking column subtraction (Math-magic: Book 3 2006). Change also appears as an important theme in these textbooks. The quantitative relation between two varying quantities is discussed at several places: the weight of a growing child which, according to a traditional custom, determines the weight of sweets distributed on her birthday (Grade 3), the number of elders in each generation of a family tree, the annual growth of a rabbit population, the growth chart of a plant over a number of days (all in Grade 5, Math-magic: Book 5 2008). No letter symbols are used in these examples, and relationships are expressed in terms of numerical tables, diagrams or charts.

These strands in the NCERT primary school textbooks are not taken up and developed further in the NCERT middle school textbooks, which appear to begin afresh by introducing a symbolic approach to algebra. A part of the reason lies in the fact that the two sets of textbooks are produced by different teams, and the schedule of publication does not always allow for smooth co-ordination. (The grade 6 textbook, for example, was published two years before the grade 5 textbook.) Another reason, we hypothesize, is the pressure to build students' capabilities with symbolic algebra, which is needed for secondary school mathematics. Curriculum design involves striking a balance between different imperatives. The balance realized in the primary mathematics textbooks emphasizes immersion in realistic contexts, concrete activities, and communicating the view that mathematics is not a finished product (Mukherjee 2010, p. 14). The middle school curriculum is more responsive to the features of mathematics as a discipline and emphasizes the abstract nature of the subject. In the words of the coordinator for the middle school textbooks, "learners have to move away from these concrete scaffolds and be able to deal with mathematical entities as abstract ideas that do not lend themselves to concrete representations" (Dewan 2010, p. 19f).

Besides finding ways of building on the strands of algebraic thinking that are present in the primary curriculum and textbooks, a concern, perhaps even more pressing in the curriculum design context in India, is to find more effective ways for the majority of children to make the transition to the symbolic mathematics of secondary school. Algebra underlies a large part of secondary mathematics, and many students face difficulties of the kind that are identified in studies conducted elsewhere (Kieran 2006). A compilation of common student errors from discussions with teachers includes well-known errors in simplifying algebraic expressions and operating with negative numbers (Pradhan and Mavlankar 1994). Errors involving misinterpretation of algebraic notation and of the "=" sign are common and persistent (Rajagopalan 2010). Building on students' prior knowledge and intuition to introduce symbolic algebra remains one of the challenges facing mathematics curriculum designers, and it is yet to be adequately addressed.

In this chapter, we offer a perspective on the relationship between arithmetic and algebra and an example of a teaching approach developed by a research group at the Homi Bhabha Centre led by the authors to manage the transition from arithmetic to symbolic algebra. The key aspect of this approach is focusing on symbolic arithmetic as a preparation for algebra. Students work with numerical expressions, that is, expressions without letter variables, with the goal of building on the operational sense acquired through the experience of arithmetic. This, however, requires a shift in the way expressions are interpreted. The aim is not just to compute the value of an expression, but to understand the structure of the expressions. Numerical expressions offer a way of expressing the intuitions that children have about arithmetic and have the potential to strengthen this intuition and enhance computational efficiency. To enable this transition, numerical expressions must be viewed not merely as encoding instructions to carry out a sequence of binary operations, but as revealing a particular operational composition of a number, which is its "value." Thus facility with symbolic expressions is more than facility with syntactic transformations of expressions; it includes a grasp of how quantities or numbers combine to produce the resultant quantity. This view of expressions leads to flexibility in evaluating expressions and to developing a feel for how transforming an expression affects its value. We argue that understanding and learning to "see" the operational composition encoded by numerical expressions is important for algebraic insight. We elaborate on the notion of operational composition in a later section and discuss how this perspective informs a teaching approach developed through trials with several batches of students.

The idea that numerical expressions can capture students' operational sense or relational thinking has been explored in other studies (for example, Fujii and Stephens 2001). In appropriate contexts, students show a generalized interpretation of numbers in a numerical expression, treating them as quasi-variables. We will review these findings briefly in a later section. The idea that algebra can enhance arithmetical insight is a view that finds support in the Indian historical tradition of mathematics. Algebra is viewed not so much as a generalization of arithmetic, but rather as providing a foundation for arithmetic. An implication is that building on the arithmetic understanding of students is, at the same time, looking at symbols with new "algebra eyes." It is not widely known that Indian mathematicians achieved impressive results in algebra from the early centuries CE to almost modern times. The fact that Indian numerals and arithmetic were recognized as being superior and adopted first by the Islamic cultures and later by Europe is more widely known. The advances in arithmetic and algebra are possibly not unconnected, since arithmetic may be viewed as choosing a representation of the operational composition of a number in a way that makes calculation easy and convenient. In the next section, we shall briefly review some of the achievements in Indian algebra and discuss how the relation between arithmetic and algebra was viewed in the Indian historical tradition.

Arithmetic and Algebra in the Indian Mathematical Tradition

India had a long standing indigenous mathematical tradition that was active from at least the first millennium BCE till roughly the eighteenth century CE, when it was displaced by Western mathematics (Plofker 2009, p. viii). Some of the achievements of Indian mathematics worth highlighting are the appearance, in a text from 800 BCE, of geometrical constructions and statements found in Euclid's *Elements*, including the earliest explicit statement of the "Pythagoras theorem," discussion of the binomial coefficients and the Fibonacci series in a work dated to between 500 and 800 CE, the solution of linear and quadratic indeterminate equations in integers, a complete integer solution of indeterminate equations of the form $x^2 - Ny^2 = 1$ ("Pell's" equation) in a twelfth century text, the finite difference equation for the sine function in the fifth century CE and power series expansion for the inverse tangent function in the fourteenth century CE (Mumford 2010; Plofker 2009).

It is well attested that Indian numerals and arithmetic were adopted first by the Islamic civilization following exchanges between the two cultures around the eighth century CE, and later by Europe (Plofker 2009, p. 255). Indian algebra was also developed by this time as seen in the seventh century work of Brahmagupta, which we shall discuss below. However there are important differences between Arabic algebra (as found in al-Khwarizmi's work *al-jabr*, for example) and the algebra in Indian mathematical texts. Two of the main differences are that Arabic algebra avoided negative quantities, while Indian texts routinely used them, and Indian algebra used notational features such as tabular proto-equations and syllabic abbreviations for unknown quantities, while Arabic algebra was purely rhetorical (Plofker 2009, p. 258f).

We will first give an overview of how topics in arithmetic and algebra are organized in the central texts of Indian mathematics, and then turn to explicit statements about the relation between arithmetic and algebra. Indian texts containing mathematics from the first millennium CE are typically one or more chapters of a work dealing with astronomy. Purely mathematical texts appear only later, as for example, in the work of Bhaskara II in the twelfth century CE. The *Aryabhateeyam,* from the 5^{th} century CE, one of the oldest and most influential astronomical-mathematical texts, contains a single chapter on mathematics that includes arithmetic and the solution of equations.

The *Brahmasphuta Siddhanta* (c. 628 CE) by Brahmagupta, considered to be one of the greatest Indian mathematicians of the classical period, has two separate chapters dealing respectively with what we might classify as arithmetic and algebra. The word that Brahmagupta uses for the second of these chapters (algebra) is *kuttaka ganita* or "computation using *kuttaka*." *Kuttaka* (frequently translated as "pulverizer") is an algorithm for reducing the terms of an indeterminate equation, which is essentially a recasting of the Euclidean algorithm for obtaining the greatest common divisor of two natural numbers (Katz 1998). Interestingly, puzzles called *kuttaka* are found even now in folklore in India and require one to find positive integer solutions of indeterminate equations. (For an example, see Bose 2009.)

The "arithmetic" chapter in the *Brahmasphuta siddhanta* deals with topics such as the manipulation of fractions, the algorithm for cube roots, proportion problems of different kinds and the "rule of three" (a representation of four quantities in proportion with one of them unknown), the summation of arithmetic progressions and other kinds of series, miscellaneous computational tips, and problems dealing with geometry and geometrical measurement (Colebrooke 1817). The *kuttaka* or algebra chapter deals with techniques for solving a variety of equations. In the initial verses of this chapter, we find the oldest extant systematic description in the Indian tradition of rules of operating with various kinds of quantities: rules for operations with positive and negative quantities, zero, surds (irrational square roots of natural numbers), and unknown quantities. The approach of beginning the discussion of algebra by presenting the rules of operations with different kinds of numbers or quantities became a model for later texts. Laying out these rules at the beginning prepared the way for demonstrating results and justifying the procedures used to solve equations.

Mathematicians who came after Brahmagupta referred to algebra as *avyakta ganita* or arithmetic of unknown quantities, as opposed to *vyakta ganita* (arithmetic of known quantities). Others, starting from around the 9th century CE, have used the word *bijaganita* for algebra. *Bija* means "seed" or "element," and *bijaganita* has been translated as "computation with the seed or unknown quantity, which yields the fruit or *phala*, the known quantity (Plofker 2007, p. 467). The word *bija* has also been translated as "analysis" and *bijaganita* as "calculation on the basis of analysis" (Datta and Singh 1938/2001). "*Bijaganita*" is the word currently used in many Indian languages for school algebra.

Bhaskara II from the 12th century CE (the numeral "II" is used to distinguish him from Bhaskara I of an earlier period) devoted two separate works to arithmetic and algebra—the *Lilavati* and the *Bijaganita*, respectively, both of which became canonical mathematical texts in the Indian tradition. Through several remarks spread through the text, Bhaskara emphasizes that *bijaganita,* or analysis, consists of mathematical insight and not merely computation with symbols. Bhaskara appears to have thought of *bijaganita* as insightful analysis aided by symbols.

> Analysis (*bija*) is certainly the innate intellect assisted by the various symbols [*varna* or colors, which are the usual symbols for unknowns], which, for the instruction of duller intellects, has been expounded by the ancient sages… (Colebrooke 1817, verse 174)

At various points in his work, Bhaskara discourages his readers from using symbols for unknowns when the problem can be solved by arithmetic reasoning such as using proportionality. Thus after using such arithmetic reasoning to solve a problem involving a sum loaned in two parts at two different interest rates, he comments, "This is rightly solved by the understanding alone; what occasion was there for putting a sign of an unknown quantity? … Neither does analysis consist of symbols, nor are the several sorts of it analysis. Sagacity alone is the chief analysis …" (Colebrooke 1817, verse 110)

In response to a question that he himself raises, "if (unknown quantities) are to be discovered by intelligence alone what then is the need of analysis?", he says, "Because intelligence alone is the real analysis; symbols are its help" and goes on to repeat the idea that symbols are helpful to less agile intellects (*ibid.*).

Bhaskara is speaking here of intelligence or a kind of insight that underlies the procedures used to solve equations. Although he does not explicitly describe what the insight is about, we may assume that what are relevant in the context are the relationships among quantities that are represented verbally and through symbols. We shall later try to flesh out what one may mean by an understanding of quantitative relationships in the context of symbols.

The word "symbol" here is a translation for the sanskrit word *varna*, meaning color. This is a standard way of representing an unknown quantity in the Indian tradition—different unknowns are represented by different colors (Plofker 2009, p. 230). Bhaskara's and Brahmagupta's texts are in verse form with prose commentary interspersed and do not contain symbols as used in modern mathematics. This does not imply, however, that a symbolic form of writing mathematics was not present. Indeed, in the Bakshaali manuscript, which is dated to between the eighth and the twelfth centuries CE, one finds symbols for numerals, operation signs, fractions, negative quantities and equations laid out in tabular formats, and their form is closer to the symbolic language familiar to us. For examples of the fairly complex expressions that were represented in this way, see Datta and Singh (1938/2001, p. 13).

Bhaskara II also explicitly comments about the relation between algebra and arithmetic at different places in both the *Lilavati* and the *Bijaganita*. At the beginning of the *Bijaganita,* he says, "The science of calculation with unknowns is the source of the science of calculation with knowns." This may seem to be the opposite of what we commonly understand: that the rules of algebra are a generalization of the rules of arithmetic. However, Bhaskara clearly thought of algebra as providing the basis and the foundation for arithmetic, or calculation with "determinate" symbols. This may explain why algebra texts begin by laying down the rules for operations with various quantities, erecting a foundation for the ensuing analysis required for the solution of equations as well as for computation in arithmetic. Algebra possibly provides a foundation for arithmetic in an additional sense. The decimal positional value representation is only one of the many possible representations of numbers, chosen for computational efficiency. Algebra may be viewed as a tool to explore the potential of this form of representation and hence as a means to discover more efficient algorithms in arithmetic, as well as to explore other convenient representations for more complex problems.

At another point in the *Bijaganita,* Bhaskara says, "Mathematicians have declared algebra to be computation attended with demonstration: else there would be no distinction between arithmetic and algebra" (Colebrooke 1817, verse 214). This statement appears following a twofold demonstration, using first geometry and then symbols, of the rule to obtain integer solutions to the equation $axy = bx + cy + d$. Demonstration of mathematical results in Indian works often took geometric or algebraic form (Srinivas 2008), with both the forms sometimes presented one after the other. The role of algebra in demonstration also emerges when we compare the discussion of quadratic equations in the arithmetic text *Lilavati* and the algebra text *Bijaganita*. In the *Lilavati,* the rule is simply stated and applied to different types of problems, while in the *Bijaganita,* we find a rationale including a reference to the method of completing the square.

Algebra in earlier historical periods has often been characterized as dealing with "the solution of equations" (Katz 2001). While this view is undoubtedly correct in a broad sense, it is partial and misses out on important aspects of how Indian mathematicians in the past thought about algebra. Most importantly, they laid stress on understanding and insight into quantitative relationships. The symbols of algebra are an aid to such understanding. Algebra is the foundation for arithmetic and not just the generalization of arithmetic, implying that arithmetic itself must be viewed with "algebra eyes." Further, algebra involves taking a different attitude or stance with respect to computation and the solution of problems; it is not mere description of solution, but demonstration and justification. Mathematical insight into quantitative relationships, combined with an attitude of justification or demonstration, leads to the uncovering of powerful ways of solving complex problems and equations.

Making procedures of calculation more efficient and more accurate was often one of the goals of mathematics in the Indian tradition, and the discovery of efficient formulas for complex and difficult computations in astronomy was a praiseworthy achievement that enhanced the reputation of mathematicians. Thus not only do we find a great variety of procedures for simple arithmetic computations, but also for interpolation of data and approximations of series (Datta and Singh 1938/2001). The *karana* texts contain many examples of efficient algorithms (Plofker 2009, pp. 105ff). In the "Kerala school" of mathematics, which flourished in Southern India between the 14th and the 18th centuries CE, we find, amongst many remarkable advancements including elements of calculus, a rich variety of results in finding rational approximations to infinite series. Thus algebra was related as much to strengthening and enriching arithmetic and the simplification of complex computation as to the solution of equations. It was viewed both as a domain where the rationales for computations were grasped and as a furnace where new computational techniques were forged.

From a pedagogical point of view, understanding and explaining why an interesting computational procedure works is a potential entry point into algebra. Since arithmetic is a part of universal education, a perspective that views algebra as deepening the understanding of arithmetic has social validity. Thus, while algebra builds on students' understanding of arithmetic, in turn, it reinterprets and strengthens this understanding. In the remaining sections, we explore what this might mean for a teaching learning approach that emphasizes the arithmetic-algebra connection.

Building on Students' Understanding of Arithmetic

Modern school algebra relies on a more extensive and technical symbolic apparatus than the algebra of the *Bijaganita*. As students learn to manipulate variables, terms, and expressions as if they were objects, it is easy for them to lose sight of the fact that the symbols are about quantities. In the context of arithmetic, students have only learned to use symbols to notate numbers and to encode binary operations, usually carried out one at a time. Algebra not only introduces new symbols such as letters and expressions, but also new ways of dealing with symbols. Without guidance from intuition, students face great difficulty in adjusting to the new symbolism. So Bhaskara's precept that algebra is about insight into quantities and their relationships and not just the use of symbols is perhaps even more relevant to the learning of modern school algebra.

What do students carry over from their experience of arithmetic that can be useful in the learning of algebra? Do students obtain insight into quantitative relationships of the kind that Bhaskara is possibly referring to through their experience of arithmetic, which can be used as a starting point for an entry into symbolic algebra? Of course, one cannot expect such insight to be sophisticated. We should also expect that students may not be able to symbolize their insight about quantitative relationships because of their limited experience of symbols in the context of arithmetic.

Fujii and Stephens (2001) found evidence of what they call students' relational understanding of numbers and operations in the context of arithmetic tasks. In a missing number sentence like $746 + \_\_ - 262 = 747$, students could find the number in the blank without calculation. They were able to anticipate the results of operating with numbers by finding relations among the operands. Similar tasks have also been used by others in the primary grades (Van den Heuvel-Panhuizen 1996). Missing number sentences of this kind are different from those of the kind $13 + 5 = \_\_\_ + 8$, where the algebraic element is limited to the meaning of the "=" sign as a relation that "balances" both sides. Relational understanding as revealed in the responses to the former kind of sentence lies in anticipating the result of operations without actual calculation. Fujii and Stephens argue that in these tasks although students are working with specific numbers, they are attending to general aspects by treating the numbers as "quasi-variables."

Students' relational understanding, as described by Fujii and Stephens, is a form of operational sense (Slavit 1999), limited perhaps to specific combinations of numbers. The students' performance on these tasks needs to be contrasted with the findings of other studies. For example, Chaiklin and Lesgold (1984) found that without recourse to computation, students were unable to judge whether or not $685 - 492 + 947$ and $947 + 492 - 685$ are equivalent. Students are not consistent in the way they parse expressions containing multiple operation signs. It is possible that they are not even aware of the requirement that every numerical expression must have a unique value. It is likely, therefore, that students' relational understanding are elicited in certain contexts, while difficulties with the symbolism overpowers such understanding in other contexts. Can their incipient relational understanding develop into a more powerful and general understanding of quantitative relationships that can form the basis for algebraic understanding, as suggested by Bhaskara? For this to be possible, one needs to build an idea of how symbolization can be guided by such understanding, and can in turn develop it into a more powerful form of understanding. In a later study, Fujii and Stephens (2008) explored students' abilities to generalize and symbolize relational understanding. They used students' awareness of computational shortcuts (to take away six, take away ten and add four) and developed tasks that involved generalizing such procedures and using symbolic expressions to represent them.

Other efforts to build students' understanding of symbolism on the basis of their knowledge of arithmetic have taken what one may describe as an inductive approach, with the actual process of calculation supported by using a calculator (Liebenberg et al. 1999; Malara and Iaderosa 1999). In these studies, students worked with numerical expressions with the aim of developing an understanding of the structure by applying operation precedence rules and using the calculator to check their computation. These efforts were not successful in leading to an understanding of structure that could then be used to deal with algebraic expressions because of over-reliance on computation (Liebenberg et al.) or because of interpreting numerical and algebraic expressions in different ways (Malara and Ioderosa). The findings suggest that an approach where structure is focused more centrally and is used to support a range of tasks including evaluation of expressions, as well as comparison and transformation of expressions, may be more effective in building a more robust understanding of symbolic expressions.

We attempted to develop such an approach in a study conducted as a design experiment with grade 6 students during the two-year period 2003–2005. The teaching-learning approach evolved over five trials, with modifications made at the end of each trial based on students' understanding as revealed through a variety of tasks and our own understanding of the phenomena. The first year of the study consisted of two pilot trials. In the second year, we followed 31 students over three teaching trials. These students were from low and medium socio-economic backgrounds, one group studying in the vernacular language and one in the English language. Each trial consisted of $1\frac{1}{2}$ hours of instruction each day for 11–15 days. These three trials, which comprised the main study were held at the beginning (MST-I), middle (MST-II), and end of the year (MST-III) during vacation periods. The schools in which the students were studying followed the syllabus and textbooks prescribed by the State government, which prescribe the teaching of evaluation and simplification of arithmetic and algebraic expressions in school in grade 6 in a traditional fashion—using precedence rules for arithmetic expressions and the distributive property for algebraic expressions. Discussion with students and a review of their notebooks showed that only the vernacular language school actually taught simplification of algebraic expressions in class 6; the English school omitted the chapter.

These students joined the program at the end of their grade 5 examinations and were followed till they completed grade 6. They were randomly selected for the first main study trial from a list of volunteers who had responded to our invitation to participate in the program. The students were taught in two groups, in the vernacular and the English language respectively by members of the research team.

Data was collected through pre- and post-tests in each trial, interviews conducted eight weeks after MST-II (14 students) and 16 weeks after MST-III (17 students), video recording of the classes and interviews, teachers' log and coding of daily worksheets. The pre- and post-tests contained tasks requiring students to evaluate numerical expressions and simplify algebraic expressions, to compare expressions without recourse to calculation and to judge which transformed expressions were equivalent to a given expression. There were also tasks where they could use algebra to represent and draw inferences about a given problem context, such as a pattern or a puzzle. In the written tests, the students were requested to show their work for the tasks. The students chosen for the interview after MST-II had scores in the tests which were below the group average, at the average, and above the group average, and who had contributed actively to the classroom discussions. The same students also participated in the interviews after MST-III, and a few additional students were also interviewed. The interviews probed their understanding more deeply, using tasks similar to the post test. In particular, they probed whether student responses were mechanical and procedure-based or were based on understanding.

The overall goal of the design experiment was to evolve an approach to learning beginning algebra that used students' arithmetic intuition as a starting point. The specific goal was to develop an understanding of symbolic expressions together with the understanding of quantitative relationships embodied in the expressions. Although this was done with both numerical and algebraic expressions, the approach entailed more elaborate work with numerical expressions by students compared to the approach in their textbooks. Students worked on tasks of evaluating expressions, of comparing expressions without calculation, and of transforming expressions in addition to a number of context-based problems where they had to generalize or explain a pattern. A framework was developed that allowed students to use a common set of concepts and procedures for both numerical and algebraic expressions. Details of the study are available in Banerjee (2008a). Here we shall briefly indicate how the teaching approach evolved, describe the framework informing the approach, and present some instances of students' responses to the tasks.

In the pilot study, students worked on tasks adapted from Van den Heuvel-Panhuizen (1996), and that were similar to the tasks used by Fujii and Stephens (2001). We found several instances of relational thinking similar to those reported by Fujii and Stephens. For example, students could judge whether expressions like $27 + 32$ and $29 + 30$ were equivalent and also give verbal explanations. One of the explanations used a compensation strategy: "the two expressions are equal because we have [in the first expression] taken 2 from 32 and given it to 27 [to obtain the second expression]." Students worked with a variety of such expressions, containing both addition and subtraction operations, with one number remaining the same or both numbers changed in a compensating or non-compensating manner (Subramaniam 2004). Some pairs were equivalent, and some pairs were not. For the pairs which were not equivalent, they had to judge which was greater and by how much. As seen in the explanation just cited, students used interesting strategies including some ad-hoc symbolism, but this did not always work. In general, when they attempted to compare the expressions by merely looking at their structure and not by computing, students made accurate judgements for expressions containing the addition operation but not for those containing the subtraction operation. Similarly they were not always successful in judging which expression was greater in a pair of expressions when the compensation strategy showed that they were unequal.

We noticed that students were separating out and comparing the additive units in the pairs of expressions but were comparing numbers and operation signs in inconsistent ways. This led to an approach that called attention more clearly to additive units in comparison tasks. However, an important moment in the evolution of the approach was the decision to use a structure-based approach for comparing as well as for evaluating numerical expressions. Other important aspects of the approach were dealing with arithmetic and algebraic expressions in a similar manner in the different tasks and relating these processes to algebraic contexts of generalizing and justification of patterns. We have described the evolution of the approach in greater detail elsewhere (Banerjee and Subramaniam submitted). Here we describe a framework that supports a structure based approach to working with numerical expressions on a range of tasks including evaluation, comparison and transformation.

The Arithmetic-Algebra Connection—A Framework

As we remarked earlier, learning algebra involves learning to read and use symbols in new ways. These new ways of interpreting symbols need to build on and amplify students' intuition about quantitative relationships. The view that algebra is the foundation of arithmetic, held by Indian mathematicians, entails that students need to interpret the familiar symbols of arithmetic also in new ways. The literature on the transition from arithmetic to algebra has identified some of the differences in the way symbols are used in arithmetic and algebra: the use of letter symbols, the changed interpretation of key symbols such as the "=" sign, and the acceptance of unclosed expressions as appropriate representations not only for operations but also for the result of operations (Kieran 2006). An aspect related to the last of the changes mentioned that we wish to emphasize is the interpretation of numerical and algebraic expressions as encoding the operational composition of a number.

The use of expressions to stand for quantities is related to the fact that, while in arithmetic one represents and thinks about one binary operation, in algebra we need to represent and think about more than one binary operation taken together. As students learn computation with numbers in arithmetic, they typically carry out a single binary operation at a time. Even if a problem requires multiple operations, these are carried out singly in a sequence. Consequently, the symbolic representations that students typically use in arithmetic problem-solving contexts are expressions encoding a single binary operation. In the case of formulas, the representation may involve more than one binary operation, but they are still interpreted as recipes for carrying out single binary operations one at a time. They do not involve attending to the structure of expressions or manipulating the expressions. Indeed, one of the key differences of the arithmetic approach to solving problems, as opposed to the algebraic, is that students compute intermediate quantities in closed numerical form rather than leaving them as symbols that can be operated upon. And these intermediate quantities need to be thought about explicitly and must be meaningful in themselves (Stacey and Macgregor 2000).

The representational capabilities of students need to be expanded beyond the ability to represent single binary operations before they move on to algebra. In the traditional curriculum, this is sought to be achieved by including a topic on arithmetic or numerical expressions, where students learn to evaluate expressions encoding multiple binary operations. However, students' work on this topic in the traditional curriculum is largely procedural, and students fail to develop a sense of the structure of expressions. As discussed earlier, students show relational understanding in certain contexts, but in general have difficulty in interpreting symbolic expressions.

One problem that arises when numerical expressions encode multiple binary operations is that such expressions are ambiguous with respect to operation precedence when brackets are not used. At the same time, one cannot fully disambiguate the expression using brackets since the excessive use of brackets distracts from the structure of the expression and is hence counter-productive. Students are, therefore, taught to disambiguate the expression by using the operation precedence rules. The rationale for this, namely, that numerical expressions have a unique value is often left implicit and not fully grasped by many students. Even if the requirement is made explicit, students are unlikely to appreciate why such a requirement is necessary. The transformation rules of algebra are possible only when algebraic expressions yield numerical expressions with a unique value when variables are appropriately substituted. Thus disambiguating numerical expressions is a pre-condition for the use of rules of transformations that preserve the unique value of the expression. Since students are yet to work with transformations of expressions, they cannot appreciate the requirement that numerical expressions must be unambiguous with regard to value.

In the traditional curriculum, students' work with numerical expressions is limited and is seen merely as preparatory to work with algebraic expressions. How does one motivate a context for work with numerical expressions encoding multiple binary operations? Student tasks with such expressions need to include three interrelated aspects—representational, procedural (evaluation of expressions), and transformational. To fully elaborate these aspects, we need to interpret expressions in a way different from the usual interpretation of an expression as encoding a sequence of such operations to be carried out one after another, a sequence determined by the visual layout in combination with the precedence rules. The alternative interpretation that students need to internalize is that such expressions express or represent the *operational composition* of a quantity or number. In other words, the expression reveals how the number or quantity that is represented is built up from other numbers and quantities using the familiar operations on numbers. This interpretation embodies a more explicit reification of operations and has a greater potential to make connections between symbols and their semantic referents. The idea of the operational composition of a number, we suggest, is one of the key ideas marking the transition from arithmetic to algebra.

Let us illustrate this idea with a few examples: (i) the expression $500 - 500 \times 20/100$ may indicate that the net price is equal to the marked price less the discount, which in turn is a fraction of the marked price, (ii) the expression $5 \times 100 + 3 \times 10 + 6$ shows the operational composition expressed by the canonical representation of a number (536) as composed of multiunits which are different powers of ten, (iii) the expression $300 + 0.6t$ may indicate cell phone charges as including a fixed rent and airtime charges at a fixed rate per unit of airtime. In examples (i) and (iii), the operational composition refers back to quantities identifiable in particular situations, while in example (ii) abstract quantities are put together or "operationally composed" to yield the number 536. It is important to preserve both these senses in unpacking the notion of operational composition.

By operational composition of a quantity, we mean information contained in the expression such as the following: what are the additive part quantities that a quantity is composed of? Are any of these parts scaled up or down? By how much? Are they obtained as a product or quotient of other quantities? The symbolic expression that denotes the quantity simultaneously reveals its operational composition, and in particular, the additive part quantities are indicated by the *terms* of the expression.

A refined understanding of operational composition includes accurate judgments about relational and transformational aspects. What is the relative contribution of each part quantity (each term) as indicated by the expression? Do they increase or decrease the target quantity? Which contributions are large, which small? How will these contributions change if the numbers involved change? How does the target quantity change when the additive terms are inverted, that is, replaced by the additive inverse of the given term? What changes invert the quantity as a whole? What are the transformations that keep the target quantity unchanged? If additive parts are themselves composed from other quantities, how do we represent and understand this?

The idea of the operational composition encoded by an expression is similar to the idea of a function but is more general and less precise. Looking at an expression as a function has a more narrow focus: how does the target quantity vary when one or more specific part quantities are varied in a systematic manner while retaining the form of the operational composition? When expressions are compared and judged to be equivalent, we judge that different operational compositions yield the same value. However, the idea of operational composition may play a role in developing the understanding of functions.

When we interpret expressions as encoding operational composition, we are not restricted to algebraic expressions. In fact, numerical expressions emerge as an important domain for reasoning about quantity, about relations and transformations, and for developing a structure based understanding of symbolic representation through the notion of operational composition. The pedagogical work possible in the domain of numerical expressions as a preparation for algebra expands beyond what is conceived in the traditional curriculum. Numerical expressions emerge as a domain for reasoning and for developing an understanding of the structure of symbolic representation.

When students' tasks focus on numerical expressions as encoding operational composition, attention is drawn to the relations encoded by the expression. Students are freed from the need to unpack the expression as a sequence of operations, fixed by a set of operation precedence rules. In the teaching approach that we developed, we emphasized ways of working with expressions that attend to the structure of expressions and are broadly aimed at developing an insight into quantitative relationships that must accompany working with symbols.

A simple numerical expression like $5+3$ is usually interpreted as encoding an instruction to carry out the addition operation on the numbers 5 and 3. In changing the focus to operational composition, the first transition that students make is to see the expression as "expressing" some information about the number 8. This information can be expressed verbally in various ways: 8 is the sum of 5 and 3, 8 is 3 more than 5, etc. Other expressions such as $6+2$ or 2×4 contain other information about the number 8, i.e., they encode different operational compositions of the number 8. Starting from this point, students move on to expressions with two or more operations of addition and subtraction. Each expression gives information about the number which is the "value" of the expression, and reveals a particular operational composition of the number.

What grounding concepts can scaffold students' attempts to study and understand the operational composition revealed in an expression? The basic level of information is contained in the *terms* or the additive units of the expression. Simple terms are just numbers together with the preceding "+" or "−" sign. Positive terms increase the value of the number denoted by the expression and negative terms decrease the value. Additive units are dimensionally "homogenous," and can be combined in any order.

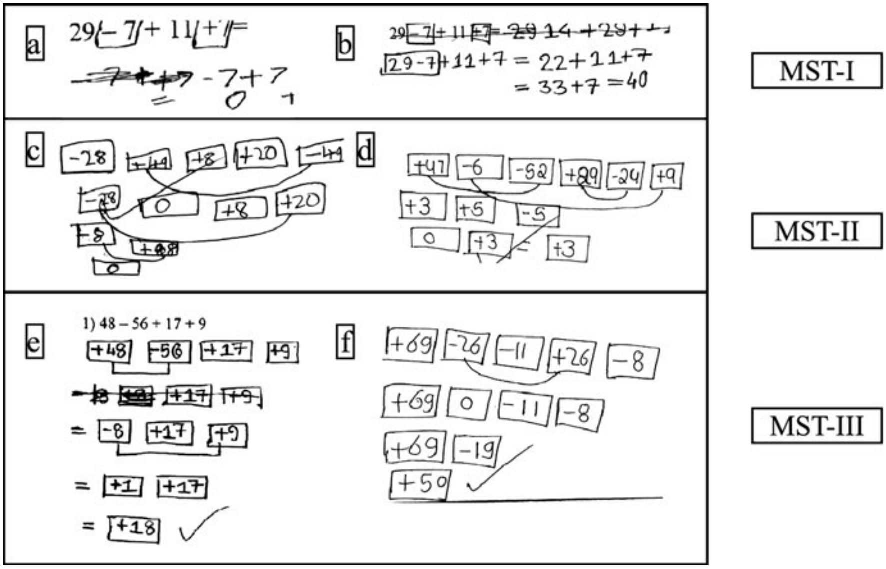

Fig. 1 Evaluation of expressions containing only simple terms by students using flexible ways in the three trials of the study (MST I, II and III)

This shift in perspective subtly turns attention away from procedure towards structure. In order to evaluate an expression, students do not need to work out and implement a sequence of binary operations in the correct order. Rather, to determine the value of the expression, they may combine simple terms in any order, keeping in view the compensating contributions of positive and negative terms. The concept of negative terms provides an entry point into signed numbers as encoding increase or decrease, which is one of the three interpetations of integers proposed by Vergnaud cited in Fuson (1992, p. 247). The approach of combining simple terms in any order, affords flexibility in evaluating an expression or in comparing expressions that is critical to uncovering structure. Thus students may cancel out terms that are additive inverses of one another; they may gather together some or all of the positive terms or the negative terms and find easy ways to compute the value of the expression by combining terms. Figure 1 shows students combining terms in flexible ways while evaluating expressions rather than proceeding according to operation precedence rules. Since the identification of additive units namely, terms, is the starting point of this approach, we have described this approach elsewhere as the "terms approach" (Subramaniam 2004; Banerjee and Subramaniam 2008).

Identifying the additive units correctly is one of the major hurdles that some students face. This is indicated by the frequency of such errors as "detachment of the minus sign" ($50 - 10 + 10 = 30$), and "jumping off with the posterior operation" ($115 - n + 9 = 106 - n$ or $106 + n$) (Linchevski and Livneh 1999). Although these errors are often not taken to be serious, they are widespread among students and impede progress in algebra. Not having a secure idea about the units in an expression and not knowing how they combine to produce the value may enhance the experience of algebra as consisting of arbitrary rules.

In working with transformations of expressions, some studies indicate that visual patterns are often more salient to students than the rules that the students may know for transforming expressions (Kirshner and Awtry 2004), suggesting that visual routines are easier to learn and implement than verbal rules. One advantage with the "terms approach" is the emphasis on visual routines rather than on verbal rules in parsing and evaluating an expression. Terms were identified in our teaching approach by enclosing them in boxes. In fact, the rule that multiplication precedes addition can be recast to be consistent with visual routines. This is done by moving beyond *simple terms*, which are pure numbers with the attached $+$ or $-$ sign, to *product terms*. In expressions containing "$+$," "$-$," and the "$\times$" operation signs, students learn to distinguish the product terms from the simple terms: the product terms contain the "$\times$" sign. Thus in the expression $5 + 3 \times 2$ the terms are $+5$ and $+3 \times 2$. In analyzing the operational composition encoded by the expression, or in combining terms to find the value of the expression, students first identify the simple and the product terms by enclosing them in boxes. The convention followed is that product terms must be converted to simple terms before they can be combined with other simple terms. Thus the conventional rule that in the absence of brackets multiplication precedes addition or subtraction is recast in terms of the visual layout and operational composition. Product terms are the first of the complex terms that students learn. Complex terms include product terms, bracket terms (e.g., $+(8-2\times 3)$) and variable terms (e.g., $-3 \times x$).

The approach included both procedurally oriented tasks such as evaluation of expressions and more structurally oriented tasks, such as identifying equivalent expressions and comparing expressions. As remarked earlier, one of the main features of the approach evolved only after the initial trials—the use of the idea of terms in the context of both procedurally and structurally oriented tasks. In the earlier trials, the use of the idea was restricted to structurally oriented tasks involving comparison of expressions, and the operation precedence rules were used for the more procedurally oriented tasks of evaluating expressions. By using the "terms idea" in both kinds of tasks, students began to attend to operational composition for both evaluating and comparing expressions, which allowed them to develop a more robust understanding of the structure of expressions. By supporting the use of structure for the range of tasks, this approach actually blurred the distinction between structural and procedural tasks. Students' written as well as interview responses revealed that they were relatively consistent in parsing an expression and that they appreciated the fact that evaluation of a numerical expression leads to a unique value (Banerjee 2008a; Banerjee and Subramaniam submitted).

In the students' written responses, we found a reduction as they moved from the first trial (MST-I) to the last (MST-III) in the common syntactic errors in evaluating numerical expressions or in simplifying algebraic expressions such as the conjoining error ($5 + x = 5x$), the detachment error described above, and the LR error (evaluating an expression from left to right and ignoring multiplication precedence). More importantly, students who were interviewed showed a reliance on identifying simple and complex terms to assess whether a particular way of combining terms was correct. Their understanding of procedural aspects was robust in the sense that they were able to identify and correct errors in a confident manner, when probed with alternative ways of computing expressions.

The interviews also revealed how some students were able to use their understanding of terms to judge whether two expressions were equal. One of the questions required students to identify which expression was numerically greater, when two expressions were judged to be unequal. Although this was not a question familiar to the students from classroom work, they were able to interpret the units or terms in the expression to make correct judgments. The following interview excerpt post-MST III from one of the better performing students illustrates how the idea of operational composition could be put to use in making comparisons:

> Interviewer: Ok. If I put $m = 2$ in this first expression [$13 \times m - 7 - 8 \times 4 + m$] and I put $m = 2$ in the original expression [$13 \times m - 7 - 8 \times m + 4$], would I get the same value?
>
> *BK: No.*
>
> Interviewer: It will not be. Why?
>
> *BK: Because it is* 8×4 *[in the first expression], if it [the value of m] is 4 here, then it would be the same value for both.*
>
> [The student is comparing the terms which are close but not equal: -8×4 and $-8 \times m$. She says that if m were equal to 4, the expressions will be equal, but not otherwise.]
>
> Interviewer: … If I put $m = 2$ in (this) expression [$-7 + 4 + 13 \times m - m \times 8$] and $m = 2$ in the original expression [$13 \times m - 7 - 8 \times m + 4$], then would they be the same?
>
> *BK: Yes.*
>
> Interviewer: Why?
>
> *BK: Because, m is any number, if we put any number for that then they would be the same.*
>
> [Comparing the two expressions the student judges correctly that they are equal.]

Our study focused largely on expressions that encoded additive composition and, to a limited extent, combined it with multiplicative composition. Learning to parse the additive units in an expression is an initial tool in understanding the operational composition encoded by the expression. Multiplicative composition as encoded in a numerical expression is conceptually and notationally more difficult and requires that students understand the fraction notation for division and its use in representing multiplication and division together. In our study, multiplicative composition was not explored beyond the representation of the multiplication of two integers since students' understanding of the fraction notation was thought to be inadequate.

Even with this restriction, the study revealed much about students' ability to grasp operational composition and showed how this can lead to meaningful work with expressions as we have tried to indicate in our brief descriptions above. It is generally recognized that working with expressions containing brackets is harder for students. While this was not again explored in great detail in the study, we could find instances where students could use and interpret brackets in a meaningful way. In an open-ended classroom task where students had to find as many expressions as they could that were equivalent to a given expression, a common strategy was to replace one of the terms in the given expression, by an expression that revealed it as a sum or a difference. For example, for the expression, $8 \times x + 12 + 6 \times x$, students wrote the equivalent expression $(10-2) \times x + 12 + (7-1) \times x$, using brackets to show which numbers were substituted. This was a notation followed commonly by students for several such examples. Besides the use of brackets, this illustrates students using the idea that equals can be substituted one for the other, and that "unclosed" expressions could be substituted for "closed" ones. In the same task, students also used brackets to indicate use of the distributive property as for example, when they wrote for the given expression $11 \times 4 - 21 + 7 \times 4$ the equivalent expression $4 \times (7 + 11) - 21$.

The study also included work with variable terms and explored how students were able to carry over their understanding of numerical expressions to algebraic expressions. We found that students were capable of making judgments about equivalent expressions or of simplifying expressions containing letter symbols just as they were in working with numerical expressions. This did not, however, necessarily mean that they appreciated the use of algebraic symbols in contexts of generalization and justification (Banerjee 2008a). The culture of generalization that algebra signals probably develops over a long period as students use algebraic methods for increasingly complex problems.

We have attempted here to develop a framework to understand the arithmetic-algebra connection from a pedagogical point of view and to sketch briefly how a teaching approach informed by this framework might begin work with symbolic algebra by using students' arithmetic intuition as a starting point. Although the design experiment through which the teaching approach was developed was not directly inspired by the historical tradition of Indian mathematics, we have found there a source for clarifying the ideas and the framework that underlie the teaching approach. The view that understanding quantitative relationships is more important than just using symbols and the idea that algebra provides the foundation for arithmetic are powerful ideas whose implications we have tried to spell out. We have argued that symbolic expressions, in the first instance, numerical expressions, need to be seen as encoding operational composition of a number or quantity rather than as a set of instructions to carry out operations. We have also pointed to the importance, from a perspective that emphasizes structure, of working with numerical expressions as a preparation for beginning symbolic algebra.

References

ASER report (2010). *Annual Status of Education Report* (*Rural*) *2009* (*Provisional*). New Delhi: ASER Centre. Retrieved from http://asercentre.org/asersurvey/aser09/pdfdata/aser09.pdf.

Banerjee, R. (2008a). *Developing a learning sequence for transiting from arithmetic to elementary algebra*. Unpublished doctoral dissertation. Mumbai: Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research.

Banerjee, R. (2008b). Assessing the curriculum reforms in India: the case of integers and algebra for beginning middle school students. In *International Conference on Mathematics Education* (*ICME-11*), Mexico. Retrieved from http://dg.icme11.org/document/get/99.

Banerjee, R., & Subramaniam, K. (2008). Bridging arithmetic and algebra: Evolution of a teaching sequence. In O. Figueras et al. (Eds.), *International Group of the Psychology of Mathematics Education: Proceedings of the Joint Meeting of PME 32 and PME-NA XXX (PME29)* (Vol. 2, pp. 121–128). Morelia, Mexico.

Banerjee, R., & Subramaniam, K. (submitted). Evolution of a teaching approach for beginning algebra.

Bose, A. (2009). Mathematical riddles among the Mushars: Linked to a historical tradition. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd of the International Group for the Psychology of Mathematics Education* (Vol. 5, p. 439). Thessaloniki, Greece.

Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Ng, S. F., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades: A cross-cultural comparative perspective. *Zentralblatt fuer Didaktik der Mathematik (International Review on Mathematics Education)*, *37*(1), 5–15.

Chaiklin, S., & Lesgold, S. (1984). *Prealgebra students' knowledge of algebraic tasks with arithmetic expressions*. Paper presented at the annual meeting of the American Research Association.

Colebrooke, H. T. (1817). *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara*. London: John Murray.

Datta, B., & Singh, A. N. (1938/2001). *History of Hindu Mathematics, Vol. II*, Edition 2001. Delhi: Bharatiya Kala Prakashan.

Dewan, H. K. (2010). Pedagogy of mathematics. *Learning Curve*, *XIV*, 16–22.

Educational Initiatives and Wipro (2006). *Student learning in the Metros 2006: How well are our students learning*? Report available online from http://www.ei-india.com/full-report.pdf. Accessed 1st Nov 2009.

Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalization through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (Vol. 1, pp. 258–264). Melbourne, Australia: The University of Melbourne.

Fujii, T., & Stephens, M. (2008). Using number sentences to introduce the idea of a variable. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics, Seventieth Yearbook* (pp. 127–140). Reston, VA: NCTM.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning*. New York: MacMillan.

Katz, V. (1998). *A History of Mathematics: An Introduction* (2nd ed.). Reading, Massachusetts: Addison Wesley.

Katz, V. J. (2001). Using the history of algebra in teaching algebra. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (Vol. 2, pp. 353–359). Melbourne: University of Melbourne.

Kieran, C. (2006). Research on the learning and teaching of algebra: A broad source of meaning. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 11–49). Rotterdam, The Netherlands: Sense Publishers.

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education*, *35*(4), 224–257.

Liebenberg, R., Linchevski, L., Sasman, M. C., & Olivier, A. (1999). Focusing on the structural aspects of numerical expressions. In J. Kuiper (Ed.), *Proceedings of the 7th Annual Conference of the Southern African Association for Research in Mathematics and Science Education (SAARMSE)* (pp. 249–256). Harare, Zimbabwe.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173–196.

Malara, N., & Iaderosa, R. (1999). The interweaving of arithmetic and algebra: Some questions about syntactic and structural aspects and their teaching and learning. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. 2, pp. 159–171). Osnabrueck: Forschungsinstitut fuer Mathematikdidaktik.

Math-magic: Book 3 (2006). New Delhi: National Council of Educational Research and Training.
Math-magic: Book 5 (2008). New Delhi: National Council of Educational Research and Training.
Mathematics: Text book for class VI (2006). New Delhi: National Council of Educational Research and Training.
Mukherjee, A. (2010). The nature of mathematics and its relation to school education. *Learning curve*, *XIV*, 16–22.
Mumford, D. (2010). Review of Mathematics in India by Kim Plofker. *Notices of the American Mathematical Society*, *27*(3), 385–390.
National Centre for Educational Research and Training (2005). National Curriculum Framework. Retrieved from http://www.ncert.nic.in/html/pdf/schoolcurriculum/framework05/nf2005.pdf.
National Centre for Educational Research and Training (2006). Position paper of the National Focus Group on the Teaching of Mathematics. Retrieved from www.ncert.nic.in/html/pdf/schoolcurriculum/position_papers/math.pdf.
Plofker, K. (2007). Mathematics in India. In V. Katz (Ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook* (pp. 385–513). Princeton, New Jersey: Princeton University Press.
Plofker, K. (2009). *Mathematics in India*. Princeton, New Jersey: Princeton University Press.
Pradhan, H. C., & Mavlankar, A. T. (1994). *Compendium of Errors in Middle School Mathematics*. Mumbai: Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research.
Rajagopalan, S. (2010). Insights about student learning from an adaptive learning math program. *Learning Curve*, *XIV*, 93–99.
Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics*, *37*(3), 251–274.
Srinivas, M. D. (2008). Proofs in Indian mathematics, epilogue. In K. V. Sarma (Ed.), *Ganita-Yukti-Bhasa of Jyesthadeva, Vol. 1—Mathematics* (pp. 267–293). New Delhi: Hindustan Book Agency.
Stacey, K., & Macgregor, M. (2000). Learning the algebraic method of solving problem. *Journal of Mathematical Behavior*, *18*(2), 149–167.
Subramaniam, K. (2004). Naming practices that support reasoning about and with expressions. In *Proceedings of the International Congress on Mathematics Education (ICME 10)*, Denmark. Available online at http://www.icme10.dk/proceedings/pages/regular~pdf/RL~K~Subramanian.pdf.
Tripathi, P. (2007). Review of mathematics textbooks. *Contemporary Education Dialogue*, *4*(1), 142–151.
Van den Heuvel-Panhuizen, M. (1996). *Assessment and Realistic Mathematics Education*. Utrecht, The Netherlands: CD-ß Press Utrecht University.