Elementary Mathematics A Teaching-Learning Perspective

This article discusses the nature of the difficulty in learning and teaching elementary mathematics from the viewpoint of the psychology of learning, focusing on some key topics such as number operation and algebra and how children learn these concepts. The paper also seeks to examine the reasons why many topics in school mathematics seem difficult and to make the point that what is difficult can still be taught and learnt. What is required is often the coordination of a culturally developed sign system with an intuitive base. This coordination is a process that happens naturally and spontaneously if children are given opportunities and situations they find motivating.

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Introduction

The mathematics taught at school is relatively well-defined and constant across most contemporary cultures in the world. There is evidently agreement about its place in what we choose to pass on to younger generations. It would be a mistake, however, to equate its worth merely with its utility. Mathematics, like music, is one of the defining accomplishments of humankind, and a part of our shared legacy. But, unlike music, it has a wide, even 'unreasonably' wide application in many intellectual and practical pursuits. So, to deny access to this part of the legacy is to promote a policy of exclusion, not just in the cultural but also in the social and political sense.

The imperative of equity is in stark contrast with the actual achievement of our school students in mathematics. The state of education is dismal as a whole, but mathematics education represents its lowest point. This forces one to reflect on both the content of elementary mathematics taught in school and on its pedagogy, and on both in the context of each other. Hence the title and theme of this article. I intend in this article to discuss the nature of the difficulty in learning and teaching elementary mathematics. This will involve a focus on some of the key topics of elementary mathematics: number, operation and algebra. The viewpoint will be one of the psychology of learning: how the corresponding mental structures develop as a student acquires understanding of the key concepts of elementary mathematics.

The discussion will therefore be primarily about the insights gained from cognitive studies of mathematics learning. I will also make passing reference to the actual situation of mathematics education from time to time. In doing so, I do not wish to advocate any particular view or approach as a solution to existing problems. Rather, the purpose of these references is to not lose sight of a very pertinent background. The references will also serve as a useful counter-point to the discussion, since the meaning of elementary mathematics must be construed more broadly than is usually the case.

Mathematics in School

In a newspaper article, a teacher trainer once reported a discussion that he had with a group of teachers. He had asked the

teachers to solve the problem 981 ÷ 9. Many teachers had obtained the answer as 19 instead of 109. During the argument that ensued, teachers who had got the correct answer pointed out that one cannot get back 981 by multiplying 19 and 9. One of the teachers from the '19' group multiplied 19 and 9 and actually obtained 981, making a second error by not carrying over 8!

Clearly, the ability to merely execute a procedure does not amount to mathematical knowledge. In this distressing incident, the division procedure for many of the teachers was only an isolated prescription. School mathematics, in contrast, is a tightly integrated body of knowledge. Division is linked to multiplication as an inverse operation, and multiplication is repeated addition. As we saw, one of the teachers did attempt to invert the division operation to justify the procedure but did not really succeed in persuading his colleagues. Using the fact that multiplication is repeated addition may also not have helped in avoiding the $981 \div 9 = 19$ error. For one can always add nine 19's and make an error by failing to carry over and obtain 981 as the sum. What we need to know further is that the addition procedure is fundamentally dependent on the positional notation that we use for numbers. From another angle, division represents the process of equal sharing: how much would each worker get if we divided wages of Rs 981 among nine workers? To give only Rs 19 to each worker would surely lead to a revolt!

It is the conceptually tightly knit domain of mathematical knowledge that we want pupils to acquire in the course of school education. This is not an excessive demand on the cognitive capacities of the average student, even though the level of mathematical ability attained by most school students at present is far from this goal. We know from interaction with teachers and students that they can and do enjoy a sense of discovery in mathematics when they are exposed to it from a perspective that emphasises understanding and the connections between concepts. Such experiences serve to correct the pessimistic inferences that one is tempted to draw from the incident described above. For many teachers, the sense of discovery is genuine since the exposure is different from what they are used to in the course of their teaching. One begins to think that the problem of mathematics education is the manner in which mathematics is taught and learnt generation after generation. School mathematics,

it would seem, is in the grip of a teaching tradition with a misplaced emphasis.

One must emphasise that developing an understanding of mathematics even at the school level is not trivial. The widespread belief among teachers and students that mathematics is difficult attests to this. How then can one claim that it is not beyond the cognitive capacity of the average student? It is in answering this question that a closer analysis of how children learn elementary mathematical concepts is of relevance. We shall see that from the cognitive point of view, even the simple concepts that are acquired spontaneously by nearly every child represent impressive achievements. The coordination of separately learnt concepts (or schemas) that is required for the concept of number to develop, is not qualitatively different from the coordination required to acquire other concepts in school mathematics. The purpose of this article is therefore twofold – to examine the reasons why many topics in school mathematics seem difficult, and to persuade the reader that what is difficult can still be taught and learnt.

Developing an understanding of the structure of mathematics is not the only element missing in the received tradition of mathematics teaching. Mathematics is also an excellent training ground in problem solving. The importance of problem solving ability in an increasingly technologically-driven economy is not difficult to concede. In mathematics, a great variety of welldefined problems can be posed. The difficulty level of the problems can be finely controlled to satisfy pedagogic specification. However, one may legitimately raise the question whether there is a general domain-independent problem solving ability and whether acquiring the ability to solve problems in one domain, say mathematics, transfers to other domains. In problem solving, knowledge and technique are not everything. Some of the most important factors in developing problem solving ability are attitudes and beliefs about one's own abilities [Schoenfeld 1987]. Even if the techniques that one learns in the course of problem solving in mathematics cannot be applied in other domains, the attitude and the willingness to tackle problems are likely to be carried over. Some research has addressed the question of transfer, but the transfer of skills and attitudes developed over a long term across broadly defined domains is a difficult question to investigate. Nevertheless, the possibility of transfer is an assumption that underlies much of the effort in education. If we wish students to benefit from learning problem solving, both cognitive and attitudinal factors need to be addressed squarely and emphasis must be laid on solving new or unfamiliar problems.1

Mathematics occupies a special place in our cultural heritage since it exemplifies knowledge that has been rigorously examined for its validity. Mathematics rests on the solid foundation of a strict notion of deductive proof. It is for this reason that since Greek times, mathematics has spurred advances in other branches of knowledge: astronomy, philosophy, logic, physics and chemistry. The axiomatic structure into which geometry was cast by Euclid represented something worth emulating in a wide variety of disciplines.

One might think that deductive proof and the axiomatic structure of mathematics are not of much concern to school mathematics, which is an enterprise aimed at educating the wider citizenry and public. A cursory comparison of the school curriculum with the mathematics taught in college shows that while proofs are central to college mathematics, they are only barely present in school arithmetic and algebra. It is only in the synthetic

geometry taught at school that proofs have an important place. One might think that this is addressed only to a fraction of the student population, namely, those who will eventually take up a study of mathematics or a related subject. However, deductive reasoning and the presentation of topics organised in an axiomatic structure address a need felt by many students who experience the freedom to question what they are learning. When their thinking is unfettered, children naturally ask for justification of the propositions that are thrown at them. As a teacher attempts to answer them, such demands for justification can regress rapidly to the simpler propositions that underlie others, but which are harder to prove or justify. If the process of inquiry is not halted by dogmatic injunction, it will only be satisfied by a properly developed axiomatic structure. The opportunity to experience this movement of thought is offered by the deductive organisation of mathematics. The axiomatic structure also provides a sequencing of the elements of knowledge that has a reasonable basis and can hence be reconstructed by a learner. This makes it possible for the piece of knowledge to be recalled and rescanned without the aid of external representations, an activity that is important if learning is to result in genuine understanding.

Cognitive studies of mathematics learning have focused a great deal on issues of understanding core mathematical concepts. The research literature on problem solving or on the role of deductive reasoning in school mathematics is relatively sparse. This is not an indication that the importance of these aspects is not recognised. Rather, the widespread failure of mathematics teaching in creating a mathematically literate population, together with the impetus from the traditional discipline of cognitive psychology have given mathematics education its current focus. Accordingly, in the sections below, I will not discuss problem solving and deductive reasoning further but will focus on how much we have understood of how children learn about numbers, operations and algebra.

The Understanding of Number

The most fundamental concept of mathematics is number. How is the knowledge of number structured in the developing mind of the child? It is important to remember that in the psychological context, we view knowledge from the perspective of learning. The tools and symbolism that we use to structure mathematical knowledge for other purposes may therefore be inappropriate in understanding mathematical cognition. Formally, the concept of number can be defined axiomatically, starting with Peano's axioms for natural numbers and defining other kinds of numbers in terms of the natural numbers: negative numbers as additive inverses of natural numbers, rational numbers as equivalence classes of ordered pairs of integers, real numbers as dedekind cuts and so on. From the psychological point of view, to have the concept of the natural number is to acquire a stable perception of the invariance of a set of objects under certain kinds of transformations. We shall elaborate this view, due to Piaget, below.

It would be appropriate to mention here the approach that Piaget used to understand the structure of knowledge in human cognitive agents. His approach was to study the successive stages of knowledge as it develops in children, to describe the differences among these stages with respect to each other and with respect to fully developed adult knowledge. This 'genetic epistemological' approach, as Piaget termed it, allowed Piaget to identify

characteristic and unforeseen differences between children's knowledge and adult knowledge. Some of Piaget's findings have defined the agenda for much of the later work in cognitive developmental psychology.

One of the most striking experiments that Piaget did had to do with children's understanding of number. In the number conservation task, children were shown two rows of objects, both rows containing an equal number of objects so arranged that the one-to-one correspondence between the rows was evident. When the children were asked whether the two rows contained the same number of objects, the children judged them to be equal. Then the experimenter spread out the objects in one of the rows while the children were looking on and repeated the question. Surprisingly, many young children, typically those younger than six years of age, said that the longer row contained more objects although nothing had been added to or removed from the row.

While this experimental result has been replicated a large number of times, the interpretation of the result has generated a significant debate [Bryant 1996]. Why do the children fail to see that the numbers are still equal in the two rows? One of the explanations is that the children misinterpret the implicit social dimension of the task. The experimenter does something, produces a visible change and repeats a question that was asked earlier. Under such circumstances, children might feel compelled to change their answer, especially since one can expect them to be unclear about the experimenter's intentions. Variations of the task have been devised to sidestep this difficulty: the children are asked only one question after the equal rows are presented and one row is altered, or a 'naughty Teddy' is introduced who deliberately disturbs the arrangement to confuse the experimenter. These variations do reduce the number of children failing the task, but the surprising failure of younger children and the systematic development with age that Piaget found persist.

A different response to Piaget's discovery of the failure to conserve number by young children was made by Gelman and her colleagues, whose work has had much influence on early number learning research. In a widely cited study, Gelman and Gallistel (1978) interpreted the failure in the conservation task as a result of the children's poor skills at counting. For Piaget, one-to-one correspondence between two sets is the defining relation for cardinality and is hence the foundation for the concept of number. The act of establishing one-to-one correspondence represents the basic schema for making a judgment about cardinality.² This need not however be the case. Establishing oneto-one correspondence between sets is hard when the sets have an arbitrary spatial layout, which is how children encounter most collections in real life. Children may be therefore be disinclined to apply such a difficult procedure in any situation. In contrast to Piaget, for Gelman, the primary schema for establishing the equality of two sets having the same cardinality is the action of counting. Her claim is that till children have fully developed their skills at counting, they are not confident of their judgments about relative numerosities and hence fail conservation tasks.

Accordingly, Gelman's approach redefines the central questions in understanding how children acquire the concept of number. She asks what the basis is on which children build their skill of counting. Do they possess an understanding of the principles underlying the act of counting at a very young age? She identifies five such principles.

(1) One-to-one correspondence: This principle is framed differently from Piaget's one-to-one correspondence. For Piaget, the

correspondence was between two sets, while for Gelman the primary correspondence relevant to counting is the one-to-one correspondence between the tokens used for counting (the number words) and the items that are tagged by the tokens. Each and every item in the set must be tagged once and once only.

- (2) Stable ordering: The tokens used for counting, that is, to tag the elements of the set must have a fixed order. The order may be different from the conventional order of number words, and sometimes is different for young children who are beginning to count
- (3) *Item indifference:* All items are equal with respect to the tagging action. Homogeneous or heterogeneous sets can be counted. (4) *Order irrelevance:* The order in which different items are tagged has no bearing on the outcome of the counting.
- (5) Cardinality: The last token called out (the last number word, for example) names a property of the entire set. Thus if the last tag is '5', 5 is the property of the set as a whole, namely, its cardinal value.

Gelman's significant finding is that children's behaviour shows the operation of these counting principles from a very young age, from about four years or even younger. This is in sharp contrast to Piaget, who did not recognise children as having the concept of number till six or seven years of age. If very young children are indeed aware of these principles, then they are not just playing a 'word game' when they count or pretend to count. It is highly improbable that they internalise the 'how to count' principles by merely observing adults count without having some internal 'template' that allows them to implicitly extract the principles. The line of argument is very much Chomskyian and is an instance of the considerable influence of Chomsky's work on syntax on the whole field of cognitive developmental psychology. Gelman's response to Piaget rests on the competence-performance distinction. Children have a basic competence – the internalisation of the how-to-count principles – but have not yet perfected their skill at actual counting. Hence their reluctance to apply it to tasks such as number conservation. When faced with a situation that requires them to compare the numerosity of two sets, children need to rely on some characteristic in order to make a judgment. If both counting and one-to-one correspondence are judged to be unreliable from past experience, they are likely to rely on a characteristic such as length, thereby producing the failure elicited in the Piagetian conservation experiment.

Bryant (1996) argues that Gelman's criteria are too weak to test the presence of the concept of number; knowing the counting principles is not the same as having the concept of number. For Piaget, and for Bryant, both cardinality and ordinality are essential to the concept of number. Cardinality is the basis for judging two sets to be equally numerous. Ordinality allows children to conclude that a set of cardinality 4 is bigger than a set of cardinality 3, and is hence bigger than a set of cardinality 2. Thus 2, 3 and 4 are not just number words which appear in a fixed order, but indicate relative sizes. In Bryant's view, Gelman does not consider these issues to be central. For Gelman, knowing the principle of cardinality is merely to recognise that the last number counted is a property of the set as a whole. The Piagetian criteria say that the child must know more, that two sets having the same cardinality are equal in number and vice versa. Some children who have a stable knowledge of the counting principles in fact fail to reach this conclusion about equal sets. Greco (1962, cited in Bryant 1996) found that children who correctly counted two rows of objects and said that each row contained five objects,

still judged one of the rows to be more numerous on the basis of length cues. In another experiment, Sophian (1988, cited in Bryant 1996) asked children to judge whether a puppet who had been given a task requiring counting was doing the right thing. Sometimes the puppet was asked to find how many objects were in front of it altogether and sometimes it was asked to compare two sets. Younger children failed to notice that the puppet made mistakes, especially when it counted all the objects when it was asked to compare two sets. Thus, although these children were able to count, in their case counting did not seem to be related to quantity. Thus we have a result that seems to go against Gelman's view that children fail number conservation tasks because their lack of counting skills prevents them from applying counting to judgments about numerosity.

Does this mean support for Piaget's view then? Do these children fail the number conservation tasks because they have no idea of one-to-one correspondence, or if they do, they do not link it to quantity? In an interesting experiment, Frydman and Bryant (1988, cited in Bryant 1996) had children apply one-toone correspondence in a sharing situation. Children are familiar with the 'one for you', 'one for me' schema for sharing equally, which is a temporal version of one-to-one correspondence. The question that Frydman and Bryant asked was whether children apply this schema mechanically, without appreciating its essential link with quantity. The children had to distribute 'chocolates' (blocks) to two recipients, one of whom accepted only singles, that is, single blocks, while the other accepted only doubles, that is, double blocks. While a majority of the five-year-olds compensated while distributing the blocks so that both recipients received an equal number of chocolates, most four-year-olds failed to make an adjustment and gave one of the recipients twice the number that the other received. In a subsequent trial, colour cues were introduced by making the double block from blocks of two different colours, and having the single blocks also of the same colours. In this case, the four-year-olds not only made the required adjustment, but also carried over what they learnt and performed much better in subsequent tasks using double blocks of only one colour.

From a strictly Piagetian point of view, the finding that fouryear-old children are indeed aware of quantity and can learn to apply it flexibly in simple sharing situations is surprising. Piaget believed that a child must pass from the pre-operational to the concrete operational stage, which happens typically around six years of age, for a child to acquire the concept of number. To put it somewhat circularly, number is what is conserved when a discrete set is transformed spatially or in other ways that leaves the number invariant. To acquire the concept of number is to realise that certain transformations are irrelevant to the cardinality of a set. Piaget thought that a child can do so only when she or he is able to mentally reverse or invert the perceived transformation. Thus when we make a row of objects longer, the child imagines the row being restored to its original length and realises that as a con-sequence the number does not change. This mental reversibility of the perceived operation is a characteristic of the child in the operational stage and, according to Piaget, is beyond the abilities of the pre-operational child.

Does this imply that a pre-operational child has no understanding whatsoever of the size of a discrete set, its *numerosity*? This appears unlikely in the light of the many studies that have been done in the past few decades on infants' perception of numerosities and of the perception of numerosity by animals, including the

higher primates [Dahaene et al 1997]. Rats have been trained to discriminate numerosities of both visual (flashes) and auditory (tones) stimuli. Further, rats trained on only one kind of stimuli, either visual or auditory, spontaneously generalise and extract numerosity information from stimuli presented in the alternate mode. Pigeons have been trained to discriminate (30 per cent errors) between 45 and 50 pecks, although the ability to detect differences of five pecks is better for smaller numbers. Monkeys and chimpanzees cannot only discriminate different numerosities, but can also recognise and use the Arabic numerals for numbers from 1 to 9 and perform simple addition [Carey 2001].

Studies on infants' cognition of numerosities have shown that very young infants, even neonates, can discriminate between sets of one, two and three objects. Such studies typically follow the habituation-dishabituation experimental paradigm. Infants are exposed repeatedly to pictures showing, say, two objects varying in type, size and position till they look at these pictures for relatively constant lengths of time (habituation). They are then shown a picture of a different number of objects, say, three objects. Typically infants look longer at such pictures, and the difference in time is statistically significant, suggesting that they can discriminate the new picture from the ones that they have habituated to. Applying a similar paradigm Wynn (1992) showed that infants can also anticipate the results of adding or subtracting small numbers (1 + 1 or 2-1). In her experiment, infants were first shown an object in front of which a screen was then placed. Another object was moved behind the screen as the child looked on. When the screen was subsequently removed, infants looked longer at the 'impossible' outcomes of three objects or one object, in comparison to the 'expected' outcome of two objects. These experiments have been replicated and also controlled for various factors by other studies and do suggest that infants can discriminate the numerosities of small sets. As infants grow in the first year, their abilities expand allowing them to deal with sets up to about four objects.

How does one reconcile these various findings? From very early on, infants appear to be able to distinguish the numerosity of a set from other properties. Very young children show awareness of the how-to-count principles. But most children fail the number conservation tasks until they are about six years of age. One way to interpret these findings is to say that children have a basic ability to perceive numerosity that is probably innate. In the first few years of childhood, children learn the number sign system that has been developed by human beings over centuries. In doing so, they respect the principles underlying counting, probably drawing on their innate understanding of numerosity. But it is only when they become skilled in counting that they apply the concept of number to make judgments about numerosity. In other words, what the growing child achieves is the coordination of a culturally developed sign system with an intuitive perceptual schema for numerosity.

Piaget outlined a theory of how the growing child, starting from virtually no innate cognitive abilities, acquires the structured knowledge characteristic of the adult. Recent studies on infant and animal cognition have shown that the infant's mind is not the 'tabula rasa' that Piaget thought it was. In the domains of language, physical objects (naive physics), interaction with other humans (naive psychology), number and even with regard to living things (naive biology) infants seem to possess domain-specific innate structures of knowledge. Piaget also provided a detailed account of the stages and sub-stages that a child had to pass through in acquiring more complex structures of knowledge. These stages

were elaborated primarily in terms of the developing logicomathematical structures, which Piaget thought underlies cognition in every domain. His theory was therefore domain general. Efforts have been made to elaborate Piaget's stage theory and the succession of structures and to develop testable hypotheses on their basis. Experiments, however, have not confirmed many of Piaget's specific ideas on the sequence of development [Gelman 2000]. Indeed, it is not clear if there is a well-defined sequence that all or even most children follow. However, Piaget's discovery that the child's understanding is radically different from an adult's and many of Piaget's specific characterisations of these differences have been confirmed by subsequent research.

To acquire a stable conceptual structure in a given domain, several discrete pieces of knowledge must be brought into coordination with one another. In other words, the child must construct a coherent structure from the experiences that she has had, or from the instruction that she has been given. The perspective of *constructivis*m, that the child is an active constructor of knowledge, not a passive recipient, is one of Piaget's enduring contributions to both psychology and to education.

In the case of the concept of number, and also in other domains, constructing knowledge may involve co-ordinating an artificial, culturally developed symbolic system with an intuitive or innate base of rudimentary concepts. The sign system for numbers that children learn is the decimal positional system with place value. It is not an exaggeration to say that virtually all of the arithmetic done in primary school, till around class five, involves the mastery of the convention of the decimal system and in addition the convention for rational numbers. In comparison, there is very little in the primary school of arithmetic proper, that is, the properties of numbers as such. Let us briefly go over what is involved in the learning of school arithmetic, which mostly involves mastery of the conventions of the decimal system.

Number Operations

After the first few count words, children learn the generative rules of the number system that allows them to count larger and larger numbers. The counting numbers use the decimal structure and children need to learn its generative rules. The Indo-European family of languages have many quirks in the decade structure of the counting numbers – the 'teens' in English, the reversal of order in the two-digit number words and the words for 19, 29 ... in Hindi and Marathi. Children learning in these languages find these oddities a stumbling block in the early years of schooling. Although they eventually master the generative rules and the counting structure, the difficulty that they face initially can hamper their learning of the rules and procedures for the basic operations. The convention of the number system is then brought in coordination with the basic understanding of the conservation of quantity and the properties of addition and subtraction. Next, a sizeable knowledge base of addition and subtraction facts is learnt. In building these facts and in extending their capability in addition and subtraction, children bring to bear their understanding of the conventional aspects of the number system and the structure of the operations to develop flexible procedures to add and subtract. Of course, children develop such a flexible operational sense only when they have the opportunity to do so. Many schools do not emphasise 'mental mathematics'. Exercises in mental computation, where children are encouraged to solve problems in different ways, are extremely important in building sound conceptual knowledge and confidence in using numbers. It is important to realise that explicitly teaching children procedures for mental computations is of limited value. As children build up this conceptual base, their skills add up to an impressive achievement and many children attain these levels with apparently little explicit instruction.

Evidence for this claim is obtained from the studies of 'street mathematics'. It is not uncommon to find many adults employing non-standard procedures for computation during commercial transactions. These procedures are often employed flexibly, reflecting a robust conceptual understanding of the basic operations. Such knowledge is found not only among adults but also among children. Studies conducted with children involved in street vending in Brazil showed them to be using flexible mental calculation strategies to find the cost of multiple items. When problems involving the same numbers and operations were posed in the formal style of school word problems, the same children made repeated errors. Clearly, the children were constructing their knowledge of 'street mathematics' independent of the school [Carraher et al 1980].

It seems inconsistent then that many children do so poorly at certain procedures, for example, the subtraction algorithm or the division algorithm. One must note that learning the algorithm for the basic operations is quite different from acquiring an operational sense. The algorithms are simplified procedures, which reduce the operations to steps calling for operations on single-digit numbers. The decimal positional notation makes this reduction possible. Of course, special attention must be paid when the single-digit operations are non-standard, for example, when subtracting a larger digit from a smaller digit, or when zero is involved. It is here that many children make errors. Many of them mistakenly assume that they are dealing with only single digit numbers. Evidently, for these children, the algorithm is completely dissociated from their conceptual understanding.

An interesting study in the late 1970s by Brown and VanLehn (1980) attempted to build, using a computer programme, a cognitive model of the erroneous knowledge of the subtraction procedure that many students have. Many of the errors that children made could be reconstructed from purely procedural considerations. Children who did not remember the procedure for special cases, such as when you need to borrow from a zero in the next place, made a 'repair' by adopting an alternative procedure, leading to the errors observed. The model, which accounted for a large percentage of the errors made by students, is further evidence of the dissociation between procedural and conceptual knowledge.

The fact that such dissociation is widespread has prompted some researchers to emphasise the construction by children of their own procedures for addition and subtraction and to withhold the teaching of standard algorithms till children construct such procedures [Fuson et al 1997]. It is not clear whether this approach will be acceptable to the wider community of teachers and curriculum designers. After all, the algorithms taught at school are simple and general and hence ex-tremely powerful. One needs to devote sufficient time to allow children to develop a mastery of these algorithms. Of late, there is a growing appreciation of how a conceptual understanding of the place value system and flexible procedures for the operations can be combined with teaching the standard algorithm to enhance students' understanding [Ma 1999].

To sum up, children eventually acquire a mastery of the counting numbers and the operations of addition and subtraction. Not much

more than just exposure to the culturally developed number sign system is required for this to happen. Indeed, not every child who grows up to be an adult may be able to carry out the school procedures of addition and subtraction without making errors. But their knowledge of these operations is sufficiently robust and flexible for them to evolve their own strategies of solving problems in situations that matter to them. The research connected with beginning mathematics has led some researchers to conclude that there is an innate basis for the development of knowledge about numbers and the basic operations of addition and subtraction [Gelman 2000]. What then of the mathematics that lies beyond this basic core?

Beyond Natural Numbers

The topics in school mathematics that lie beyond natural numbers and the operations of addition and subtraction are rarely learnt spontaneously by children as they grow into adulthood. These are the topics that pose hurdles to children in school. Gelman (2000) and others suggest that a robust knowledge of natural numbers can, in fact, interfere with the learning of other topics, such as, for example, rational numbers. At least some of the difficulties that children face in various topics of school mathematics are due precisely to their success in understanding and internalising the structure of the counting numbers and the operations of addition and subtraction.

The difficulties that children have with rational numbers illustrates this point.³ The notation for a rational number is introduced in primary school through the concept of a fraction. Some of the most common errors that children make when dealing with fractions and decimal numbers are due to the erroneous extension of concepts from their knowledge of whole numbers. For example, 1/8 is thought to be bigger than 1/7, and 8.19 is thought to be bigger than 8.7. Children have a great difficulty in understanding the magnitude of a fraction or of a decimal fraction. It has been suggested that in order to understand fractions, children need to develop the schema of subitising or flexibly changing the unit used to measure out a quantity. For example, a collection of 12 can be seen in different ways, as 12 ones, or as 6 twos, or as 4 threes and so on. This schema also forms the basis for what is called multiplicative thinking, where it is important to understand the multiplicative relation between numbers, that is, their ratio. We use fractions in order to quantify a magnitude that is not a whole number but lies between whole numbers. In specifying a fraction, as we will elaborate below, we choose a unit that is smaller than one to measure out the magnitude. For decimal fractions the smaller units are always powers of 1/10.

Thus if we are measuring a length that is more than 6 units and less than 7 units, and we wish to assign a number to the length, we proceed by dividing the interval between 6 and 7 into 10 equal parts, assigning the numbers 6.1, 6.2, 6.3, ... to the divisions in the interval. If this is not accurate enough, say, we find that the length is between 6.2 and 6.3, then we divide the interval between 6.2 and 6.3 into ten equal parts, marking the divisions as 6.21, 6.22, 6.23 This procedure can be carried out indefinitely to any accuracy that we desire, generating successive digits after the decimal point.

The case of ordinary fractions is more complicated than that of decimal fractions because the intervals between whole numbers are divided into a varying number of equal parts: into halves, thirds, fourths and so on. Each way of dividing produces a different unit – the so-called unit fractions 1/2, 1/3, 1/4 ... and so on. The unit fractions can be ordered, and as the denominator increases, the unit fraction becomes smaller. Every fraction is either a unit fraction or is composed of unit fractions. For example, 3/5 or three-fifths, is equal to 1/5 + 1/5 + 1/5. If children are to develop a sense of the magnitude of a fraction they need to grasp the notion of a unit fraction thoroughly and understand the relation between a given fraction and the unit fraction from which it is composed.⁴ To take a simple example, if we were to compare 3/5 and 3/7, we see that they are both composite fractions and hence composed from unit fractions. The fraction 3/5 is made up of three-fifths or three 1/5s, while 3/7 is threesevenths or three 1/7s. Since 1/7 is a smaller fraction than 1/ 5, we conclude that 3/7 is smaller than 3/5. In other words, a fraction denotes a relation between itself and the unit interval, with the unit interval being divided into a specified number of equal parts. To grasp the relation between two fractions with different denominators requires one to see that the same unit interval can be divided in different ways into sub-units. This is analogous to what is involved when we subitise a collection of wholes, that is, when we break it in different ways using units of different sizes.

A part of the difficulty of the concept of the rational number stems from the fact that we use it in a variety of situations from which it is virtually impossible to extract the common features. We introduce fractions in the early classes, as representing partwhole relations using simple shapes such as rectangles and circles as illustrations. However, many of the basic properties of fractions - order relations of fractions, the equivalent fractions, addition and subtraction of fractions – are much clearer when we think of the fraction as a number with a definite position on the number line. As children build these concepts, we require them to simultaneously extend the application of the part-whole relationship to collections and magnitudes, for example, when we take 1/3 of 15. Fractions function as unary operators in this context. Rational numbers are also used to represent ratios of magnitudes. When we say that two lengths are in the ratio p/q, we mean that there is a unit length such that one length is 'p' times the unit length, while the other length is 'q' times the unit length. Apart from these contexts, we use the rational number notation as a convenient representation of the quotient in a division operation. For example, if 13 equal-sized rotis are shared equally among 20 children, then each child's share is precisely 13/20 of a roti. Or, if the cost of 12 pens is Rs 450, then each pen costs Rs 450/12. Children encounter all of these situations in school mathematics. One suspects that even children who successfully manage the application of rational numbers in all these situations have conveniently ignored doubts about what is common to these situations.

It is clear that children who have mastered the use of natural numbers must learn a whole new way of using numbers when dealing with fractions. Fractions do not have any order that reflects the size of the fraction. In fact, given any two fractions, one can always find a fraction that lies between them. It follows that unlike in the case of natural numbers, there is no fraction 'next' to a given fraction. Besides these critical differences, a child has to learn that the numbers in the fraction can completely mislead one, if they are thought in analogy with whole numbers. The use of large numbers in the numerator or denominator has no simple consequence for the size of a fraction. Thus 421/843

is just a little smaller than 1/2, and the fraction that is half way between 3/5 and 4/5 is 7/10. Besides these relationships, a child has to learn that there are any number of ways in which the same fraction can be written, using different whole numbers, for example 1/3, 7/21, 19/57, ... Not only that, every whole number can be written as a fraction, and in any number of ways; for example, the number 2 can be written as 4/2, 34/17, 842/421, ...

To summarise, two of the major sources of difficulty with rational numbers are the stark notational and conceptual differences of the rational number in comparison to the integers and the wide range of applications of the rational number. The notation for a rational number uses whole numbers, whose decimal structure children have managed to learn over the first few years of school. But the interpretation of the whole numbers used in the rational number notation is radically different. Further, rational numbers are used to denote part-whole relationships, to specify a part of a given collection or a magnitude, to denote ratios between two magnitudes, and as a convenient way of writing the quotient in a division operation. The power of the notation for a rational number arises from this wide range of applications. Children typically learn these various connotations of the rational number in disconnected fragments over a period of time. It is hardly surprising that many of them do not acquire even a basic understanding of the rational number. Indeed, what is surprising is that a few students do manage to use the notation correctly and confidently. Given the disconnected approach to rational numbers in most curricula, this is suggestive of the spontaneous coordination of schema acquired separately in different contexts. Rational numbers remain one of the most difficult areas of school mathematics. The pedagogic problem of how to integrate conceptual and procedural knowledge in this area, and how to facilitate the coordination of the different aspects of rational numbers in the learner has not really been solved satisfactorily.

School Algebra

We have seen how in the case of fractions, children need to learn a notation that is both sophisticated and very different from the one that they are familiar with for whole numbers. In algebra, they need to develop an entirely new way of looking at operations. Here the power of notation is raised to a new level, allowing one to solve equations, express general identities and explore functional relationships. The new language with its conventions and the higher level of abstraction pose many problems for students. Several research studies on students' understanding of algebra have been done in the past few decades and have identified many points where students have difficulty.

Such studies have attempted to build a model of students' thinking in algebra that accounts for the observed errors that students make. The early work in this area was done within the Piagetian framework. Collis (1974, cited in Kieran 1992) observed that children younger than 10 years of age had trouble with statements of the form 4+5=3+6. He inferred that students needed to put the expressions on either side of the '=' sign into a closed form in order to make sense of this equation; that is, they needed to explicitly write down the statement 4+5=9 to make the transition from the left hand side (LHS) to the right hand side (RHS). He attributed this to the children being in Piaget's concrete operational stage. Children older than 10 or 11 years cross over into the formal operational stage, when they can perceive formal relationships among symbolic expressions.

In this stage, they do not need to replace an unclosed expression with a closed form in order to perceive equivalences.

Subsequent research has moved out of the Piagetian framework, which has proved inadequate to understand the nature of students' thinking in algebra and the course of its development. It is not possible to correlate the stages in the development of students' algebraic thinking with age-related general cognitive development. Indeed, many of the errors that students make in algebra persist in later years, and even adults have considerable difficulty with algebra. In a revealing study, Clement et al (1981) asked college undergraduate students to write down an equation for the statement, 'there are six times as many students as professors' using 'S' and 'P' for the number of students and professors respectively. A surprisingly large number of students wrote the equation as 6S = P, instead of the correct equation 6P = S. The students who made the error apparently represented the syntax of the natural language sentence in the equation. We will see below that the failure to view letters as numbers and the failure to perceive the numerical relationships in expressions and equations is one of the major causes of errors in algebra. Clearly, the difficulty students have with algebra persists even when they have acquired the cognitive ability to deal with abstract objects.

A perspective that examines how algebra is structured as a symbolic system is useful in understanding algebra from the perspective of teaching and learning. The symbolic structure of school algebra can be analysed at three levels: the atomic level (analogous to words in the natural language), the level of expressions (analogous to phrases) and the level of equations (analogous to sentences). The atomic symbols are the numerals, the variables or the literal numbers and the signs for the operations. Each symbol has a clear referent: a number including an unknown or a generalised number, or an operation. In addition to these there are the grouping symbols – various kinds of brackets - which can be thought to have a syntactic function. At the next level, these atomic symbols are combined to form expressions, which have multiple semantic values. Examples of simple expressions are x + 3, 4y. The atomic symbols in the first expression are 'x', '+' and '3'; 'x' is a variable, '+' the operation sign and '3' a numeral. More complex expressions can involve several terms: $3x^2 y + y(z - x) - z(2y^2 - x)$.

The fact that an expression can have multiple meanings is pivotal to algebra and is both the source of the power of algebra and the source of difficulty for those learning it. For example, the expression x + 3 stands in the first place for a number, perhaps an unknown or a generalised number. It also stands for the result of the operation of adding 3 to x, that is, the sum of x and 3. Further, it provides information about the number that is denoted: the number is 3 more than x, or x more than 3. To illustrate these differences in meaning, consider the arithmetic expressions 5+3 and 7 + 1. Both denote the same number, namely. 8, but include different descriptions of the number. The first expression describes the number as '3 more than 5' or '5 more than 3', while the second describes it as '1 more than 7' or '7 more than 1'. Alternatively, one may think of the expressions as meaning respectively 'the sum of 5 and 3' and 'the sum of 7 and 1'. The Fregean distinction between sense and reference applies quite nicely to arithmetic or algebraic expressions.

The next symbolic level is that of equations. These are analogous to sentences. Equations are full-fledged statements that have a truth value. An equation asserts that the expression on the LHS of the equation is the same as the number denoted by the

expression on the RHS. As an illustration, first let us consider examples of arithmetic equations: 5 + 3 = 7 + 1 has the truth value 'true' since the LHS and the RHS expressions denote the same number, while the equation 6 + 2 = 7 + 3 has the truth value 'false'. Algebraic equations are more akin to truth functions, they may be true for some values of the variables and false for other values. For example, the equation x + 3 = 9 has truth value 'true' for x = 6 and 'false' for all other values of x. The set of values of the variable that make the equation true is called its *solution set*. An identity is an equation that is true for all values of the variable in a domain. Thus the equation $(x + 3)^2 = x^2 + 6x + 9$ is true for all values of x, and is therefore an identity.

The difficulties that students have with algebra may now be analysed using this framework. At the level of the variable, students appear to move through different stages as they progress in the learning of algebra [Booth 1984]. Keeping these stages in mind is useful for purposes of understanding students' errors and planning instruction. However, there is no necessity about these stages or the sequence in which they appear. At first, students may simply reject the variable that 'suddenly' appears in the arithmetic expressions that they have been dealing with. This may take the form of ignoring the variable altogether, or simply replacing it arbitrarily with a number. Students at the next stage interpret the variable as an abbreviation for an object, (for example, '6a' may be thought to mean '6 apples') or as an object in its own right and not a number. In the next stage, students correctly interpret the variable to be a number, but think of it as taking a specific value, often substituting a number that they think is appropriate. Some students even assign values based on alphabetic order: a = 1, b = 2, and so on. This is thought to be the result of an exposure to certain kinds of puzzles. At the next stage, students are ready to accept the correct notion that a variable can stand for multiple values or a range of values or can even represent a generalised number that can take any value. Here too students appear to accept this idea only gradually, and have difficulty coming to terms with the possibility that in the expression 3a +b, a and b can both take the same value.

One can also identify the broad stages of the development of students' understanding of the next level of symbolism, namely, expressions. Understanding the structure and function of algebraic expressions forms the core of algebraic thinking. So the stages that students pass through at this level are critical for the learning of algebra. Again, one must note that there is no necessity associated with these stages. The stages listed below have been adapted from Sfard and Linchevski (1994).

- Expressions are viewed as just a string of symbols. This stage is correlated with a lack of acceptance of a variable or a letter in the expression.
- Expressions are thought to be an abbreviation of a set of instructions for carrying out a computational process. Thus the expression 2x + 7 is viewed as an abbreviation for a set of instructions: take a number, multiply it by 2 and add 7. At this stage, students feel a compelling need to 'close' the expression and replace it with a number, or a 'closed' form. This is thought to be the explanation for the widespread concatenation error: substituting x + 3 with x + 3 or 3x. Presumably, students view the expression as an instruction to add 3 to x and write down the answer. Expressions are understood as standing for the result of a set of operations, that is, for a number. At this stage, students may be able to solve simple equations with the variable on only one side of the '=' sign by inverting the operations in the expression

in the reverse order. For example, they may find the unknown in the equation 2x + 5 = 17 by first subtracting 5 from 17 and then dividing the result by 2.

- Unclosed expressions are accepted and students can operate with them. Thus students may add, subtract or multiply expressions and simplify them in the course of solving a problem. They may also be able to interpret the meaning of simple expressions as discussed above.
- Different expressions can be compared and relations between expressions such as equivalence, functional relations can be understood. This is a mature stage of algebraic understanding, when students are able to understand the concept of a function.

A similar stagewise analysis may be proffered for students' understanding of the next symbolic level: that of equations. Here one of the critical transitions is from viewing the '=' sign as a signal 'to do something and produce an answer', to viewing it as signifying that the two numbers obtained on both sides are equal. Presumably, children in primary school encounter many questions of the type 7 + 6 = ?, where they need to write down an answer after the '=' sign. These experiences give rise to the interpretation of the '=' sign as a signal to produce and answer. Such children may initially be puzzled by questions like ? = 5+ 4 or 3 + 6 = 4 +?, feeling that the question has been framed wrongly. However, exposure to exercises that specifically require them to check for the equality of numbers obtained after carrying out operations on both sides is usually sufficient for children to overcome this problem. A later stage in the development of students' understanding of equations involves the appreciation of the difference between an equation and an identity. Still later is the appreciation of the rules for solving equations involving surds and polynomials.

Of these three symbolic levels, the atomic symbols, the expression and the equation, the critical level is that of the expression. Indeed, many researchers agree that this marks the difference between the arithmetic and the algebraic way of thinking. The crucial difference between algebra and arithmetic is not the presence of the 'letter' but a difference in the way in which unclosed expressions are interpreted and handled. Filloy and Rojano (1985, cited in Herscovics and Linchevski 1996) speak of a didactic cut between arithmetic and algebra. They found that students who could solve linear equations of the type 3x + 4 = 19, were not able to solve equations where the variable appeared on both sides of the '=' sign, as in, for example, the equation 5x + 2 = 2x + 11. They thought that such students were on the arithmetic side of the didactic cut, essentially because they did not view expressions as denoting a number as well as containing compositional information about the number. Sfard and Linchevski (1994) characterise the algebraic way of thinking as understanding the process-product duality of the algebraic expression: that it stands for both the process involving a set of operations and the number which is the product of that process. Several other researchers have noted the discomfort of students in operation with unclosed expressions.

The power of algebra stems from the fact that an unknown or generalised number can be represented, operated with and the result of the operation represented as an expression. The expression can again be operated upon to yield further expressions. This is nicely illustrated in the many versions of the 'guess the number' game popular with children: think of a number, add 2 to it, multiply the sum by 3. Now subtract the original number and divide the result by 2. Again subtract the original number. The

answer is 3. Writing the operations down in the form of an expression makes this obvious. Let the number be x. So we have,

$$\frac{3(x+2)-x}{2} - x = \frac{2x+6}{2} - x = 3$$

The 'guess the number' game involves the solution of equations, usually linear equations. In an equation, the letter stands for an unknown number. Other contexts require the interpretation of the letter as a generalised number. This representation turns algebra into a powerful tool for the discovery of patterns and for justifying and proving propositions. Consider the statement that when we add one to the product of consecutive odd numbers we always get a perfect square. It is quite simple to justify this using algebra. We let the two odd numbers be 2n-1 and 2n+1. Their product is $4n^2-1$. Adding 1 gives $4n^2$ which is a perfect square. In the process of justifying, we realise that the statement is true for both consecutive odd and even numbers which can simply be written as n-1 and n+1.

As the learning of algebra progresses, students need to acquire greater and greater facility with expressions. Even in the proof of the statement in the previous paragraph, it is important to choose an appropriate form of the expressions for consecutive numbers. This requires a familiarity with expressions, a sound understanding of the relations between expressions and an ability to estimate the result of operating with expressions. The prelude to all this is, of course, understanding what expressions are all about.

We have seen earlier that many children acquire flexibility with whole numbers and basic operations with minimal contribution from external instruction. This process involves coordination of the culturally developed sign system for number with innate cognitive structures that facilitate the perception of quantity. An analogous process of the coordination of a symbolic system with a base of cognitive capacity happens in case of algebra too. The base here is not an innate cognitive capacity, but rather the flexible understanding of numbers and operations that children acquire at an earlier stage. Hence mathematics has a hierarchical structure from the viewpoint of teaching and learning, and not only from the logical point of view. The signs and symbols that are mastered earlier function like semi-concrete objects for the development of the next hierarchical level. Thus numbers function as semiconcrete objects for the development of algebra. Indeed for many children, by the time they begin to learn algebra, numbers have already begun to acquire the status of concrete objects. It is this base that needs to be built upon in acquiring an understanding of algebraic expressions.

To conclude, the learning of school mathematics does not require any special ability beyond the basic cognitive capacities that most children and adults possess. What is required is often the coordination of a culturally developed sign system with an intuitive base. This coordination is a process that happens naturally and spontaneously if children are given opportunities and situations that they find motivating. The essence of the constructivist approach is that such coordination cannot be forced through instruction although it can be facilitated by a careful choice of learning tasks. There is evidence that this process can happen with whole numbers and basic operations relatively independently of instruction. But beyond this basic core, children need a teaching-learning setting in which they have opportunities to master specific bits of knowledge and to co-ordinate these fragments. As long as instruction focuses on narrow learning outcomes and treats topics in a fragmentary manner, students will not have sufficient opportunities to achieve the coordination of concepts.

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Notes

- 1 Unfortunately, the obsession with high-stakes board examinations, and the attendant social pressures have virtually reduced problem solving to merely recalling solutions to problems solved previously, often by somebody else a teacher or a textbook author. If the culture of problem solving is to flourish, activities located on the fringe of mainstream education like olympiads and olympiad-like competitions need to grow in number and in coverage.
- 2 A 'schema' for Piaget is an internal representation of a pattern of action. It is essential for Piagetian theory that schemas are applied beyond the situation in which they first appear (assimilation), that schemas change while being applied to new situations (accommodation) and that lower-level schemas are coordinated/integrated to form more complex schemas. For example, in infancy, the schema for reaching out and grasping is coordinated with the schema for purposeful looking to achieve hand-eye coordination [Piaget and Inhelder 1966].
- 3 Rational numbers are numbers which can be written in the form p/q, where p and q are integers and q is not zero. The concept of a rational number is more general than that of a fraction. The term fraction is usually restricted to the use of the rational number to specify a part-whole relation or a number that lies between whole numbers.
- 4 This is one of the omissions that is common in many textbooks prescribed for primary schools. Very few textbooks give the concept of a unit fraction the emphasis that it ought to have.

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