DESIGNING AN INSTRUCTIONAL SEQUENCE FOR TRANSITING FROM ARITHMETIC TO ALGEBRA IN THE MIDDLE SCHOOL

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Algebra has long been considered as a difficult domain by students, teachers and researchers in mathematics education. Many efforts have been made to make the algebraic activity meaningful, using both the semantics and syntax of algebra. At Homi Bhabha Centre for Science Education, we conducted a design experiment during 2003-2005 which gave support to the students in perceiving the structure of expressions and use the structure sense developed in the context of arithmetic to make a transition to algebra, giving the letter a referent (e.g. Linchevski 1995). The aim of the study was to develop a teaching sequence for beginning algebra which bridges arithmetic and algebra and builds a strong procedural and structural sense for expressions among students in contexts which deal with reasoning about and with expressions.

The teaching sequence evolved over repeated trials as part of the design experiment. Design experiments have been defined as follows:

Design experiments are extended (iterative), interventionist (innovative and design based) and theory oriented enterprises whose "theories" do real work in practical educational contexts (Cobb et al., 2003).

Design experiments were developed to carry out research for testing and refining educational designs based on theoretical principles derived from prior research (Collins et al., 2004). It entails "engineering" particular forms of learning and systematically studying the learning processes in the same context which defines and supports the learning (Cobb et al., 2003). The design is put into practice and tested and revised (progressive refinement, Collins et al., 2004) based on experience within the context of practice to lead to the development of some local domain-specific theory, addressing theoretical questions and issues delineating why it works or understand the relationships between theory, artifact and practice (Brown, 1992; Cobb et al., 2004; Collins et al., 2003). The theory intends to "identify and account for successive patterns of student thinking by relating these patterns to the means by which their development was supported and organized" (Cobb et al., 2003).

The teaching approach we conducted to connect students' arithmetic and algebra knowledge evolved over five trials between Summer 2003 and Summer 2005. The teaching approach gained in consistency and coherence, and formed a well connected structure by the end of the fourth trial. The students for the trials studied in 6^{th} grade and came from two nearby English and Marathi medium. Different groups of students were involved in the first two trials but the same group of students was involved in the last three trials. In this paper, we would describe the evolution of the teaching sequence and give rationale for choosing the tasks. We would only discuss and highlight the changes that were made in each trial based on the students' classroom responses and an analysis of their post test results. We wanted to study the development in students' understanding in a context where they were being explicitly trained to connect arithmetic and algebra.

The First Trial

The first trial was held in Summer 2003 with students who had just passed their grade 5 examinations. The main aim of the trial was to identify teaching materials which could build structure sense of arithmetic expressions among students. In order to inhibit the automatic tendency of computing an arithmetic expression by students, they were taught to look at expressions as relations and verbalize the meaning of the expressions. The students in the process learnt that an expression stands for a number which is also the value of the expression and that all the expressions for a number 'express' different information about the number. Rules for evaluating expressions were explained in the traditional way (multiplication before addition and move from left to right for operations '+' and '–'), prior to introducing tasks on developing structure sense as some minimum knowledge of syntactic manipulation was required to begin such tasks. Tasks were also created to broaden the meaning of '=' sign from the 'do something' signal to a symbol stating the relation of equivalence and these tasks were a constant feature in the trials. Students worked on exercises of filling the blank by a number so that the expressions on both sides of the '=' are equal and comparing two expressions using the signs <, =, >. The initial exercises on this issue could be done with computation. Knowing well the importance of structure sense and its role in reasoning about expressions, the latter tasks focused on comparison of expressions without computation. Students successfully completed the task and verbally explained their judgment using their intuitive knowledge of operations, as long as the expressions were positive two termed expressions (27 + 32 and 29 + 30) but faced trouble with the appearance of '–' sign in the expressions or multiple changes in the terms across the two expressions. The need was evident for some knowledge of syntactical aspects of expressions like bracket opening rules, operations on negative numbers to record and keep track of the transformations in the more complex situations. Accordingly, bracket opening rules were taught as relations of equality between the two expressions, one with bracket and the other without bracket. Subsequently, a more focused task of finding the value of an expression given the value of a related expression (van den Heuvel-Panhuizen et al., 1993) was carried out. This was an easier context than the earlier one to use transformations on expressions to reason and justify their responses and they learnt it with a little effort. Lastly, the concept of 'term' was introduced as components of an expression to delineate the surface structure of expressions. This not only helped the students parse an expression correctly but also allowed them to see relationships between the terms and with the whole expression, leading to the important idea of 'equal expressions'. It was evident to us that the concept of 'term' was very powerful and together with the concept of '=', could possibly help the students in developing structure sense and give the tools which could be used to reason about expressions. Thus, this trial not only helped us choose instructional material for building structure sense of arithmetic, but also gave us important feedback on the importance of various concepts and skills required for building this sense. Also, these concepts could be easily used in the context of algebra where flexibility in operations is essential to carry out manipulations.

The Second Trial

In the second trial (Autumn, 2003), one of our major aims was to explore how knowledge (procedural and structural) developed in the context of arithmetic could be used in the algebraic context, for which a two group experimental design was formulated. One of the groups was taught both arithmetic and algebra and the other was taught only algebra. Algebra was taught similarly to both the groups. Compared to the earlier trial, in this phase we introduced the concept of 'terms' soon after the students (who were taught both arithmetic and algebra) learnt evaluation of arithmetic expressions. This was done so as to enable students to make a connection between the procedures of evaluation (taught in a traditional manner) and the explicit parsing of expressions followed by tasks which focused on comparing expressions. Terms of an expression were categorized as 'simple term' (e.g. +3, –4) and 'product term' (e.g. +3×4, +2×y). Much of the other tasks remained the same, although the reasoning about expressions was restricted to mostly verbal statements. The concept of 'term' was used by the students in various 'comparison of expressions' tasks and 'filling the blank' tasks, sometimes spontaneously and at other times due to instruction and discussion in the classroom. Compared to the group of students who were only taught algebra, these students made some connection between procedures of evaluating expressions and their surface structure and could identify equal expressions from a list of expressions both in the context of arithmetic and algebra. Although the surface structure of arithmetic and algebraic expressions was similar and easy to grasp, the manipulation of algebraic expressions was not simple. The teaching strategy at this point failed to connect these two domains as different approaches were used for manipulation in arithmetic and algebra. Manipulation of algebraic expressions was also taught in a very traditional manner by collecting 'like terms' (product terms with same literal factor) and many students committed the conjoining error (e.g. 2+3×y=5×y). This trial indicated that arithmetic was helpful in making sense of the rules in algebra, especially those requiring the use of structure sense and meaning of expressions, but more explicit connections were needed to be built in the instructional sequence to exploit the advantage. The group which was taught only algebra learnt to simplify expressions but could not understand the meaning of those procedures as seen from their performance in the other related tasks in the post test. The concept of 'term' and 'equality' became even more important to achieve the goal. The need to bridge the gap between procedures and structure of the expressions was also evident as the students failed to connect these on their own. Also a better understanding of negative numbers was proving to be essential to carry out the various tasks successfully.

The Third Trial

In the third trial (Summer 2004), we tried to bridge the gap between procedure and structure sense by using the concept of 'term' for both the aspects. The aim was to make the teaching approach more coherent with the minimum number of distinct rules needed to carry out transformations in both the domains of arithmetic and algebra. Now the concept of 'term' was used both in the context of evaluating expressions as well as in identifying and generating equal expressions using the concepts of 'term' and 'equality'. Terms were made prominent by putting them in a box and were used to analyze expressions to decide the precedence of operations. It was assumed that this explicit use of the surface structure of expressions while evaluating arithmetic expressions would lead automatically to manipulation of algebraic expressions. This again did not succeed as the rules of manipulation were framed using different vocabulary in the two domains and students failed to see the similarity in the structure of the expressions in arithmetic and algebra and therefore the similarity in rules required to transform expressions. In the case of arithmetic, students converted product terms to simple terms and then evaluated the expression; in the case of algebra, they were combining like and unlike terms using the idea of adding and subtracting 'singletons'. To make the connection with arithmetic explicit, students evaluated algebraic expressions for certain values of the letter, which did not help much in the purpose. Negative numbers were introduced with the help of the number line and an analogous letter-number line (Carraher et al., 2001) was used to give meaning to simple algebraic expressions, like x+1, x-1. This served a dual purpose: to see the numbers and the expressions on the number line as holding relationships among each other (which they were encouraged to state verbally) as well as allow movements/ action on the number line. The 'process' and the 'object' conception (see Sfard, 1991) of these symbols was thereby consolidated. They worked on simple one step letter-number line journeys and found distance between two points on the number line. These were the contexts of reasoning with expressions. Another such context was the 'think-of-a-number' game.

The arithmetic and the algebra parts of the sequence were still very loosely connected. Different vocabularies were used in the arithmetic and the algebra parts. The concepts of 'term' and 'equality' were used in both the parts but to different extents. In the arithmetic part, these concepts had been used in all contexts but the use itself was more rule/ procedure bound rather than giving any real sense of structure which could be used flexibly in any situation. Therefore the transfer of these to the algebra context was limited, for example students did not spontaneously see any similarity in the structure of the expressions $3 + 4 \times d$ and $3 + 4 \times 2$ and students repeatedly committed the conjoining error. Students also failed to use the simplification procedures of algebraic expressions in the contexts.

The Fourth trial

These drawbacks were further improved upon in the fourth trial (Autumn, 2004). 'Term' was made more central in the whole sequence. Evaluation of expressions was no longer restricted by the rigid rules of moving from left to right. Terms were classified as simple and complex term (e.g. product and bracket term). The product term could have a numerical factor, a letter factor or a bracket factor. Simple terms could be combined easily and product terms had to be converted into simple terms and then combined with other simple terms. Combining simple terms was nothing but operations on signed numbers (positive and negative terms) which was taught using the idea of compensation of equal and opposite terms. If two product terms in an expression had the same common factor, then the product terms could be combined using the distributive property. This paved the way for integrating the procedures of simplification of arithmetic and algebraic expressions. Also this allowed the students to operate on the expressions by exploring relations between terms, rather than any fixed precedence rule. Even the bracket opening rules were reformulated using 'terms' and 'equality' in conjunction with ideas of 'inverse' and 'multiple'. The exercises on reasoning about expressions was expanded to encourage students to use their knowledge of rules and procedures to generate expressions equal to a given expression, not restricted to rearranging terms and give symbolic reasons for their judgments in 'comparing expressions' task. The structure and the procedures got connected and complemented one another as a result of this change. Each task required the use of both these aspects. This whole approach enabled us to gradually turn the processes of addition and subtraction into objects (positive and negative terms) which could be themselves manipulated. Tasks on reasoning with expressions in the context of algebra were used as earlier. Students further found patterns in cells of a calendar and represented those using letter.

At the end of the fourth trial, the arithmetic-algebra sequence had evolved to a level where the approach adopted ('Terms' approach) allowed the students to attach meaning to the operations on numbers as well as letters. It also gave them the required flexibility and opportunity to use the concepts learnt in various situations and tasks, making it into a coherent unit. The concepts of 'term' and 'equality' gave them not only visual support but also a vocabulary for communicating their reasoning about expressions. It was also realized that integer operations would be indispensable in this whole sequence. It was evident from the responses of the students that their procedure and structure sense complemented each other. There seemed to be continuity and coherence between symbolic manipulations in arithmetic and algebra as well as with regard to procedural and structural tasks. What this module lacked was continuity between symbolic algebra and using these manipulation skills in contexts requiring algebra as a tool. The students failed to spontaneously use the manipulation skills they had acquired in the contexts of representing and proving/ justifying. The time spent on these activities was also very minimal. The next trial focused on this aspect of the module.

The Fifth trial

In the final trial (Summer, 2005), the focus was on students' verbalization and articulation of various concepts and rules they had learnt in the earlier two trials and using them in situations like expressions with brackets where more than one rule was applicable or which could be solved in more than one way (e.g. $23-(4+5\times3)$). A large amount of time was spent on context-based activities in algebra requiring reasoning with expressions. They not only played the think-of-a-number game, but also generated such problems for others in the class and predicted the answer for everyone. Generalizing geometric patterns was another activity which captivated their interest and both these activities led to fruitful discussions about semantic and syntactic aspects of algebra: meaning of letters, correct representation and proper use of brackets and generalization from particular instances ('seeing the general in the particular').

CONCLUSION

The design experiment used in the research study enabled us to evolve an instructional sequence which could bridge the gap between arithmetic and algebra and tie them into a coherent unit with the minimum number of rules required to learn syntactic manipulation. Our aim was not only to make a sequence which would 'work' with the students but also to see how their understanding evolves with the changes made in the instruction and to achieve a coherence in thought, both for the teacher teaching the topic and for the students learning it. The development of the sequence gave us further insights into the difficulties students face in understanding the connection between arithmetic and algebra. Making the connection between the two domains is far from being trivial and is not spontaneously absorbed by the students. The development seen in the students' reasoning and thinking abilities at the end of the trials suggests that a radicalization of the structural approach to teaching arithmetic may lead to an understanding and appreciation of the syntax and rules of transformation in algebra.

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