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DEVELOPING PROCEDURE AND STRUCTURE SENSE OF ARITHMETIC EXPRESSIONS

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This paper describes sixth grade students' performance in tasks related to arithmetic expressions in the context of a design experiment aimed at developing a principled approach to beginning symbolic algebra. This approach, which is centered on the concept of 'term', is described elsewhere. In the paper, students' performance in two kinds of tasks over items that test procedural knowledge and items that test structural understanding is examined. We address questions of consistency in the use of procedures in different task items, and the transfer of procedural knowledge to the more structure-oriented items. The data shows that the relation between procedural knowledge and structural understanding is complex. Developing a structural understanding of expressions requires the consistent use of the procedures and rules in various situations/ contexts and making sense of the relationships between the components of the expression. We cite some preliminary evidence in favour of the effectiveness of the structure-oriented approach both in strengthening procedural knowledge and structural understanding.

BACKGROUND

A sound procedural knowledge in evaluating arithmetic expressions is clearly necessary to build a strong foundation for algebra. Manipulating algebraic expressions requires students to be well aware of the rules, properties and conventions with regard to numbers and operation signs. It has also been recognized that appreciating the structure of arithmetic expressions is useful for understanding algebraic expressions; algebra is at times described as generalized arithmetic exploiting the structure of arithmetic expressions (Bell, 1995). A poor understanding of operational laws might lead to conceptual obstacles and hinder generalizing and recognizing patterns between numbers (e.g. Williams and Cooper, 2001).

Students' experience with arithmetic expressions in traditional classrooms is mainly oriented to procedures but may be ineffective even in inducing sound procedural knowledge. Many studies have reported both the poor procedural knowledge of students and their lack of understanding of the structure of arithmetic expressions (Chaiklin and Lesgold, 1984; Kieran, 1989). Students are seen to use faulty rules of operations and are inconsistent in the way they evaluate an expression (Chaiklin and Lesgold, 1984). Many common and frequent errors are reported, such as doing addition before multiplication and detaching the numeral from the preceding negative sign (Linchevski and Livneh, 1999).

The larger project, of which this study forms a part, is aimed at developing an instructional sequence for beginning algebra that builds both sound procedural knowledge and understanding of structure of arithmetic and algebraic expressions.

FRAMEWORK ADOPTED IN THE TEACHING APPROACH

The teaching approach adopted in the project explicates the structure of arithmetic and algebraic expressions from the very beginning. It capitalizes on students' prior arithmetic knowledge and is strongly centered around the concept of term. Hence we refer to this approach as the 'terms approach' below. Here we describe briefly the way in which the term concept is used in teaching procedures and concepts. More details of this approach have been described elsewhere (Kalyansundaram and Banerjee, 2004; Subramaniam, 2004).

Students learn at the outset that an arithmetic expression stands for a number, which is the value of the expression. Two numerical expressions are equal if their values are equal. Equality of expressions can also be judged from the relationships between the components or parts of the expressions. This makes it essential for the students to learn to parse the expressions correctly, and explore and identify the relationships between the parts, and of the parts to the whole. We take structural understanding to include this group of skills. This is consistent with Kieran's (1989) definition of structure, which is seen as comprising 'surface' and 'systemic' structure.

The concept of 'term' has proved useful in this context. The concept of 'term' requires students to see the number/numeral together with its sign. Terms may be simple terms (+5) or complex terms. Complex terms can be of various types like product term (e.g. +3×2) and bracket term (e.g. -(4+2)). The product term may contain only numerical factor/s or letter factor/s or bracketed factor/s. While simple terms can be combined easily, a product term (or complex term) cannot be combined with a simple term unless the product term (or complex term) is converted into a simple term/s. Identifying the conditions when an expression remains invariant in value leads to the idea of equality of expressions. The meaning of "=" is thereby broadened from the 'do something' instruction to stand for a relation between two expressions which have the same value. The two concepts of terms and equality together give visual and conceptual support to the procedures for evaluating expressions (order of operations) and the rules for opening bracket, as they get reformulated using these two concepts.

METHODOLOGY

A design experiment methodology has been used in developing this instructional approach. The design experiment is conducted with grade 6 students (11 to 12 yr olds) from nearby English and vernacular medium (Marathi) schools. The English medium and the vernacular medium students form separate groups of instruction. The schools cater to low or mixed socio-economic strata. Four teaching intervention cycles have been conducted between summer 2003 and autumn 2004, during vacation periods of the schools.

The four teaching cycles were carried out in summer of 2003, autumn 2003, summer 2004 and autumn 2004 respectively. The first cycle was mainly exploratory in character and is not reported in this paper. There were 3 groups of students in each of the cycles 2, 3 and 4. Each group had 11 to 13 instructional sessions of 90 minutes each. A and B groups in all the cycles were from the English medium, and C groups from the Marathi medium. Subscripts indicate the cycle to which the groups belong. All the nine groups across the three cycles are discussed separately. The students in groups A_4 and C_4 were students who had attended the course in Cycle 3 except a few in C_4 who were first-timers. The students in all the groups in the previous cycles including B_4 attended the course for the first time.

Each group in a particular cycle had one teacher, except for A_2 and A_3, which had separate teachers for the arithmetic and algebra modules, who taught for about equal durations. Three teachers were involved in teaching the English groups across the cycles and one teacher for the vernacular group. Three out of the four teachers, which included the Marathi medium teacher, involved in the project were collaborators in the research project.

The details of the instruction were worked out by the group of teacher-researchers in the course of discussions held both preceding as well as during the cycles. Discussion and reflection by the group on the different teaching cycles has brought out the salient features that are common to and different in the cycles. There is an increasing centrality and coherence to the use of the concept of 'term' over the cycles. In the earlier cycles, this concept was used only in the context of judging the equality of expressions, but in the later cycles, increasingly, the procedures for evaluating expressions were brought under this concept. In terms of the evolution and coherence of the approach, cycle 4 represents the most evolved form.

The presence of multiple groups and teachers in and across the cycles helped us trace the development of students as they went through the course of instruction as well as observe the differences among them due to slight variations in the teaching sequence and their prior knowledge. It is therefore difficult to compare the groups directly. The students in Cycle 4 were exposed to the matured 'terms approach' and we will focus on their performance looking at the common errors and the extent of structural understanding. The data was collected through daily practice exercises, written tests, video-recordings, teacher's log book and the pre and the post tests given to the students. Interviews were conducted with 22 students about 6 weeks after cycle 4. The students who showed either very consistent or somewhat inconsistent knowledge of procedures and structure sense during the course were selected from the three groups for the interviews, most of them falling in the average to high category of performance. In the context of the present paper, it is important to note that groups B_2 and B_3 are slightly different in terms of the instruction received. Group B_2 received no instruction in arithmetic, but only in algebra, the extra time being spent on activities in geometry. Group B_3 received instruction mainly on arithmetic expressions that was centered around operations with signed whole numbers.

ANALYSIS OF DATA

Here we discuss the performance of the students in the pre and post tests in tasks dealing with two types of expressions: (a) expressions with a '×' and '+' sign and (b) expressions with '+' and '–' signs only. For each type of expression, we examine a set of tasks: simple tasks and complex tasks requiring essentially procedural knowledge, and tasks that require some structural understanding. The latter tasks call for judging the equality or inequality of expressions based on their structure without recourse to calculation. Since consistent interpretation of conventions used in arithmetic expressions is an essential element in building a structure sense, we examine the consistency of student responses across simple and complex procedural tasks. Specifically we look for the influence of the structure oriented teaching approach using the concept of 'terms', on consistency and on developing a structure sense.

Evaluation of expressions with a '+' and '×' sign

Many children do not absorb the convention of multiplication before addition in evaluating arithmetic expressions even after it has been taught (Linchevski and Livneh, 1999). The most common 'LR' error in evaluating expressions like 7+3×4, is to first add and then multiply, that is, to move from left to right. An earlier study conducted by us (unpublished) showed that the 'LR' error accounted for about 50% of the errors in equivalent contexts made by a group of rural upper primary teachers. Table 1 summarizes the performance of students in the different groups in evaluating an expression with a '+' and '×' sign.

Item		**Cycle 2**		**Cycle 3**		**Cycle 4**	
		Pre	Post	Pre	Post	Pre	Post
e.g. 7+3×4	A	44	88	0	74	68	93
	B	50	62	0	24	15	92
	C	23	89	21	82	74	91

N(A_2, A_3, A_4)=(25, 23, 28); N(B_2, B_3, B_4)=(21, 29, 26); N(C_2, C_3, C_4)=(34, 38, 42)

Table 2: Percentage correct in evaluating expressions with '+' and '×'

Students in the present study were not introduced to the rule of operations before class 6, which accounts for the very low rate of correct answers in the pre test of Cycle 3 for all groups in the table. Students in cycles 2 and 4 were briefly exposed to the rules of order of operation during their school instruction before they came for the vacation course. The post test results show a significant improvement in their performance in both the cycles. Also noticeable is the better performance of the students in groups A and C in the pre test of Cycle 4, the students being not only exposed to the rules in the school but also during instruction in Cycle 3. Students in group B_4 were fresh students and had only some idea of evaluating expressions from the school. The post-test scores of groups B_2 and B_3 remain low relative to the pre test and the other groups, because students received very little or no instruction on this aspect during the vacation program. While in Cycle 2, evaluating expressions was taught only as a set of rules, in Cycles 3 and 4, the 'terms approach' with increasing emphasis on the idea of product term was adopted. The incidence of LR error as a fraction of total errors in Cycles 2, 3 and 4 respectively are 6/7, 6/13 and 2/8, the remaining errors being mainly computational errors. (Groups B2 and B3, which did not receive instruction on this topic, have been excluded.)

We now examine the consistency with which students applied the '×' before '+' convention across test items. Some of the tests contained two items of the above type, one with a '+' sign and the other with a '–' sign. Students were consistent in their responses to both questions, with a few (2 to 4) answering one of the questions correctly while making the 'LR' error in the other. However, when the second item was a more complex but similar item (Cycle 2: Evaluate 3×(6+3×5)), around 17% of the students in all the groups made the 'LR' error while evaluating the expression inside the bracket although they had correctly evaluated the corresponding expression in the item without brackets.

In a related item, where a substitution was required to be done prior to evaluation (Cycles 3 and 4: $7+3\times x$, $x=2$), the students' performance was low (around 50% or lower, except for C_4 which had around 70%). Although most of the students who performed poorly on this item had a problem with substitution, a significant number of students (12%) in all the groups made the 'LR' error after substituting correctly for the variable, although they had evaluated the corresponding arithmetic expression correctly. This inconsistency on the part of the students shows that although they learnt to parse the expression correctly and had absorbed the convention of multiplication before addition and subtraction in a simpler situation, in a more complex task the 'LR' error may resurface. In Cycle 4, where the 'term' approach was adopted more strongly and the overall occurrence of 'LR' error is low, the inconsistency in the substitution question (that is, responses showing 'LR' error after substitution but not in the evaluation item) is only 7% for all the groups.

Figure 1a shows the performance of students in cycles 2 and 3 on the more structure-oriented task of judging equality for expressions of the above type. These expressions were slightly more complex than the evaluation items and had two '+' signs and one '×' sign each (therefore, two simple terms and one product term, like 28+34+21×19 or 21+34×19+28). The data indicates that knowing how to evaluate expressions of this kind is necessary but not sufficient for judging equality. Nearly all the students who can make the correct judgment about the equality/ inequality of two expressions, can also evaluate the arithmetic expression with '+' and '×' sign (See Figure 1b). The percentage of students, who can succeed in the more complex task of judging the expressions equal to a given expression, is high for the groups C_2, A_3 and C_3. In Cycle 4, the corresponding task was more complex with the options testing their ability to use brackets and splitting terms (like writing –9 as –4 –5) in the expression. We would not discuss the details of these results here.

a b

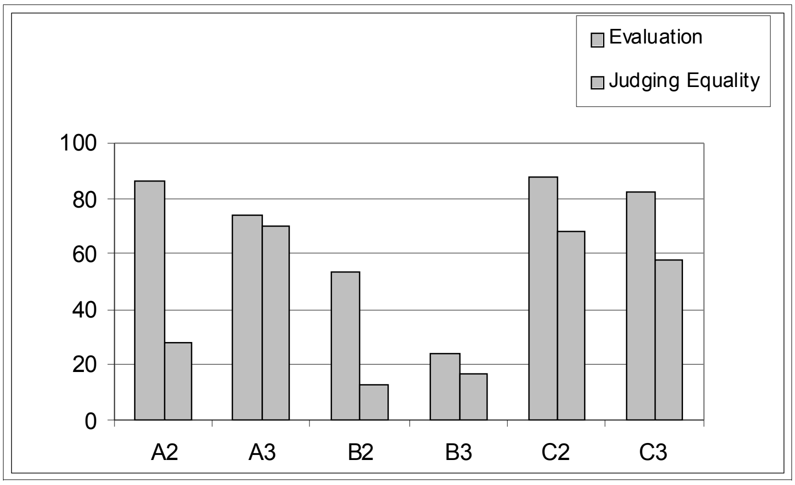

Figure 1: (a) Percentage of correct responses in evaluation and judging equality tasks for different groups. (b) Overlap of students who perform correctly on the judging equality task (indicated by the region filled with small circles) and those who perform correctly on the evaluation task (indicated by the hatched region) in all groups.

In the interviews conducted after Cycle 4, 19 students out of 22 justified their response by referring to the terms in the pair of expressions. This does not mean however that all were correct in their responses. For example, while comparing the expressions 18-15+13×4 and 4×15+18-13, 6 students identified the terms wrongly as +18, -15, +13 and ×4. This was consistent with their wrongly judging the expressions 4×15+18-13 and 18-13+15×4 as unequal. From the above, it is clear that ability to correctly evaluate simple expressions consistent with the rules of operations does not transfer readily to the more structural task of judging equality. The interview data indicate that the concept of term is readily applied to judging equality and may aid students in forming a structural understanding of expressions.

Evaluation of expressions with only '+' and '-'

An expression like 19–3+6 appears to be easy to evaluate if students know the operations of addition and subtraction. However students frequently evaluate this expression as equal to 19 – 9 = 10, making what has been described the error of detaching the negative sign (Linchevski and Livneh, 1999). In the study with teachers referred to earlier, 'detachment' errors accounted for about 40% of the errors that teachers made in equivalent contexts. One reason for this error could be incorrect perceptual parsing, where students 'detach' the minus sign from the terms to the right of the sign. Another reason, as indicated by the interview responses of some students, is that students mislearn the rule of order of operations, thinking that addition precedes subtraction. (The 'BODMAS' mnemonic actually suggests this misleading rule.) Table 2 shows the performance of students across all the cycles in evaluating this type of expression. The post test results in the even cycles is slightly better than the odd cycle, which could be due to their enhanced exposure to the evaluation task, first in school and then in our project.

Item		Cycle 2		Cycle 3		Cycle 4	
		Pre	Post	Pre	Post	Pre	Post
19-3+6 (only simple	A	64	84	61	74	68	86
	B	48	62	38	69	65	85
	C	74	94	82	74	79	86

Table 2: Percentage correct in evaluating expressions with '+' and '–'

In designing the 'terms' approach, we expected students to avoid making the detachment error as they learnt to parse an expression into terms in the course of evaluating the expression. Although the performance in the even cycles is nearly same, in Cycle 2, the rate of occurrence of the detachment error for all groups in the pre test is 31% and in the post test 17%. In this cycle, it must be recalled, the concept of term was not used in evaluation tasks but only in judging equality tasks. In the post test for Cycle 3, there are only a few cases of detachment error, the rest being mainly calculation errors, and in Cycle 4 there are no detachment errors. This supports our hypothesis concerning the effectiveness of the 'terms' approach in avoiding the detachment error.

Most of the students interviewed after Cycle 4 were confident that 25-10+5 cannot be written as 25-15. Some could not say why they thought so but others said it (i.e., 25-15) can be done only if there is a bracket around 10+5 or that the term –10 has been incorrectly changed to +10 to get 15 and added that it could be -5. These students also evaluated the expressions not in the left to right fashion but combined terms flexibly as it suited them.

The more structure-oriented tasks of judging equality for this type of expressions were specifically designed to test whether students make the detachment error. Only 20%-35% of the students made correct judgments in this type of item in Cycle 2. In the slightly simpler item in Cycle 3 (comparing expressions such as 249+165-328 or 328+165-249), 40%-60% of the students made correct judgments. The item in Cycle 4 was more difficult with a product term included in each expression and was again designed to catch the detachment error (18-27+4×6-15 & 18-20+7+4×6-10+5). Here 40% of the students made correct judgments. The fact that students were splitting the expressions into terms was corroborated in the interviews after Cycle 4. 21 out of 22 students interviewed said that the expression 49-5-37+23-5 is not equal to the expression 49-37+23 because of the extra two '–5's, but readily saw that the latter expression was equal to 49-5-37+23+5, because –5+5 gives 0.

DISCUSSION

The development of the teaching approach during the course of the project, which can be characterized as making the concept of term central to both structural (judging equality) tasks and procedural (evaluation) tasks, has proved fruitful from two points of view. Firstly, it has made the instructional approach internally coherent allowing students to deal more meaningfully with symbolic expressions. Second, it has strengthened students' procedural knowledge and has reduced the occurrence of well-known errors. Subjective assessments of the interviews conducted at the end of Cycle 4 suggest that students feel confident in the justification that they give for their responses. However, the performance in structure-oriented tasks is low even in the later cycles. This is partly due to the increased complexity of the tasks. Classroom discussions indicate that students are more confident in dealing with simpler expressions while judging equality. However, the data indicate to us that the formation of structure sense from a knowledge of procedures and rules is a difficult and long process. It would require abstracting the relationships within and between expressions. Further, it requires consistent use of the rules and procedures in various situations sharing the structural aspects.

One other consequence of our teaching approach needs to be mentioned. Identifying and comparing terms between a pair of expression in order to judge their equality is something of a shortcut in carrying out the task. When this is taught explicitly, for some students it may assume a recipe-like quality, turning what we have called a structure-oriented task to a more procedural one. In the course of the interviews, we noticed that for some students this seems to be the case, while other students develop a more flexible and truly structural understanding. This is an aspect we intend to explore further. However, even for students who interpret the 'terms approach' in recipe-like ways, we hope that the transition to an understanding of structure will be easier than in the traditional approach.

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