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Evolution of a teaching approach for beginning algebra

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Abstract The article reports aspects of the evolution of a teaching approach over repeated trials for beginning symbolic algebra. The teaching approach emphasized the structural similarity between arithmetic and algebraic expressions and aimed at supporting students in making a transition from arithmetic to beginning algebra. The study was conducted with grade 6 students over 2 years. Thirty-one students were followed for a year, and data were analysed as they participated in the three trials conducted that year. Analysis of students' written and interview responses as the approach evolved revealed the potential of the approach in creating meaning for symbolic transformations in the context of both arithmetic and algebra as well as making connections between arithmetic and symbolic algebra. Students by the end of the trials learnt to use their understanding of both procedures and a sense of structure of expressions to evaluate/simplify expressions and reason about equality/ equivalence of expressions both in the arithmetic and the algebraic contexts.

Keywords Arithmetic · Algebra · Structure · Teaching approach · Term · Equality · Expressions

1 Introduction

Early research in the area of algebra teaching and learning revealed the many difficulties that students face while learning algebra. These include students' lack of understanding of symbols/letters and of the manipulation of symbolic expressions/equations (e.g. Kuchemann, 1981; Booth, 1984; Kieran, 1992; MacGregor & Stacey, 1997). Many of these difficulties could be connected to students' poor knowledge of transformation of numerical expressions. Students could not identify equality of two arithmetic expressions without

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computation, arbitrarily computed the arithmetic expressions depending on the numbers which appeared in the expression as well as made some systematic errors, like the 'detachment error': $23-6+7=23-13=10$ (e.g. Chaiklin & Lesgold, 1984; Kieran, 1992; Linchevski & Herscovics, 1996; Linchevski & Livneh, 1999; Linchevski & Livneh, 2002). These studies indicated that students misunderstand order of operations and do not use them to develop an understanding of transformations that can keep the value of an expression equal (that is, they lack an aspect of structure sense). These ideas are important to learn in arithmetic as transformations in algebra use them as general rules and properties.[1] Algebraic expressions share structural similarity with arithmetic expressions and thus arithmetic can serve as a useful 'template' on which understanding of transformations in algebra can be built (Linchevski & Livneh, 1999).

Students' prior arithmetic exposure in computing binary operations does not prepare them enough to handle multiple operations in arithmetic expressions and to develop understanding of the crucial properties of numbers and operations. In the absence of a well-developed understanding of transformations in the numerical context, students often use arbitrary procedures to simplify algebraic expressions and may commit the same errors as in the arithmetic context (e.g. Kieran, 1992; Fischbein & Barash, 1993; Linchevski & Livneh, 1999). It is also possible that students (including in this country) develop a pseudo-structural understanding of algebraic expressions, where they may know rules of symbolic transformation and may understand two expressions as equivalent if their values are the same after replacing the letter with a number. But they may not make a connection between the two, that is, a valid transformation ought to keep two expressions equivalent and therefore for any value of the letter, expressions will lead to equal numerical value (Cerulli & Mariotti, 2001). This, to some extent, explains the sense of arbitrariness and meaninglessness that students feel while working on symbolic algebraic transformations. In order to better understand symbol manipulation in algebra, students must appreciate the following: (a) different ways of evaluating an arithmetic expression must yield the same value and (b) corresponding to these, there are valid transformations of algebraic expressions that yield expressions equivalent to one another.

Recognizing the difficulties that students face in algebraic symbol manipulation due to inadequate understanding of arithmetic expressions, we conducted a study over 2 years with grade 6 students (11–12 year olds, first year of algebra instruction) to develop a teaching approach that could help manage the transition from arithmetic to beginning symbolic algebra. The teaching approach gradually evolved through repeated trials during the study, emphasizing the structure of arithmetic expressions and using this understanding in the context of algebra. In this paper, we discuss aspects of the evolution of the teaching approach together with its potential for (1) creating meaning for symbolic transformations

[1] The words 'arithmetic' and 'algebra' are used in this article to indicate in the first place identifiable domains of the school curriculum. Algebra is typically introduced in Indian schools in grade 6 with the introduction of the use of letters. Hence, the use of 'arithmetic' or 'arithmetic expressions' or 'arithmetic context' (i.e. context of arithmetic tasks) refers to the part of the school curriculum that eschews the use of variables. Similarly, 'algebra', 'algebraic expressions' and 'algebraic tasks' refer to tasks that involve the use of letter variables and belong to an identifiable part of the school curriculum. Thus, we use the term arithmetic to include knowledge of numbers and basic operations on them, transformation of numerical expressions and knowledge of properties of operations with respect to numbers. This study uses only the integer number system. Algebra is the domain consisting of operating on and with the letter, transformation of expressions with letters, formal and generalized understanding of rules and properties of operations, and using the letter for representing, proving and generalizing. However, as much literature has shown, arithmetic tasks can embody aspects of 'algebraic thinking', and hence the 'arithmetic' tasks used in the study aim to introduce students to 'algebraic thinking' in the context of numbers.

and (2) making the transition to beginning symbolic algebra. The discussion is restricted to an arithmetic based on rules and properties of operations and numbers and an understanding of basic operations and beginning symbolic transformations in algebra, whose importance in developing a complete understanding of algebra cannot be denied. We illustrate the nature of the progress made by the students with respect to understanding and reasoning about transformational activities in arithmetic and algebra through the trials. We especially look for evidences of students' abilities to use procedures and structure of expressions in a complementary manner and to connect arithmetic and algebra.

2 Background

Many efforts have been made to use the numerical context provided by arithmetic to teach algebraic transformations at the secondary school level with varying success. For example, Booth (1984) highlighted the notational similarity between arithmetic and algebra during instruction, so as to enable students to represent simple situations using unclosed algebraic expressions but had limited success with respect to expressions with brackets. Some others like Liebenberg, Linchevski, Sasman and Olivier (1999), Liebenberg, Sasman and Olivier (1999), Livneh and Linchevski (2007), Malara and Iaderosa (1999) report studies where students were taught computations on arithmetic expressions (expressions with multiple operations, indices) in order to generalize them to the context of algebra. Except for the study by Livneh and Linchevski where the intervention was found to be helpful in transferring arithmetic learning to the algebra context in compatible situations, the other studies pointed out students' inability to generalize the properties of operations to the algebra context. Students did not use the same rules of transformation across the domains, nor did they use the rules consistently in arithmetic. Further, they could not use their understanding of computation in non-computational situations of identifying equal expressions. They continued to focus on procedures of computing arithmetic expressions and treated the symbols procedurally rather than 'proceptually' (Tall, Thomas, Davis, Gray & Simpson, 2000). In order to be successful in algebra, students need to suspend operations for a while and think about properties which can be used to simplify the algebraic expressions and thus be able to deal with the process-product duality at each step (Sfard, 1991).

These studies indicate that correct procedures of computation or correct parsing of expressions are not sufficient to make the connection between arithmetic and algebra. One way of establishing the connection is through emphasizing the structure of expressions and explicitly engaging students in discussions about rules of transformation and possibilities and constraints of transformation (e.g. Kirshner, 2001). It is this possibility that we follow in our teaching approach. A few recent efforts to develop thinking about properties of operations, numbers, relations between numbers and operations among young children in the primary grades have provided insights into their abilities to generalize and formalize in simple situations (e.g. Blanton & Kaput, 2001; Carpenter & Franke, 2001; Fujii & Stephens, 2001).

Specially designed computer environments and spreadsheets have also been found to be useful in connecting arithmetic and algebra and in making sense of algebraic representations and transformations in algebra (e.g. Thompson & Thompson 1987; Filloy, Rojano & Rubio, 2001; Ainley, Bills, Wilson & Kirsty, 2005; Chaachoua, Nicaud, Bronner & Bouhineau, 2004; Tabach & Friedlander, 2008). Even though many of these approaches (especially spreadsheets) are closely tied to numerical methods and lead to an understanding of equivalence of algebraic expressions by focusing on its denotation, there is not adequate attention paid to the structure of the expression, which includes an understanding of the units or components of the expression and how each unit affects the value of the expression. Computer intensive environments also provide the possibility to focus on more challenging aspects of algebra like analysing and organizing information, appropriate representation, and drawing valid conclusions, rather than working on only symbol manipulation. However, it is not clear if these environments can sidestep the problem of understanding symbols and symbol manipulations (Kieran, 2004).

In this study, we worked in a non-technology intensive, paper–pencil context appropriate to the situation in most schools in India. We made an effort to first develop meaning for symbols and operations among students in the context of 'transformational activities' (Kieran, 2004) and prepare them for beginning symbolic algebra. This part of the teaching approach was called 'reasoning *about* expressions' and dealt with discussing possibilities and constraints on operations in the contexts of evaluating/simplifying expressions, discussing the meaning of symbols (integers, '=', expressions, etc.), and comparing and judging equality/equivalence of expressions. It was expected that these tasks would allow the students to understand the operations and the properties associated with symbolic expressions and their transformations (for example, one can add two numbers in any order but not subtract, $a\times b$ is to be considered a unit, adding the units in any order does not change the value of the expression). In each trial, we followed up these activities with tasks based on representation, generalization, justification, and proof (e.g. pattern generalization from shapes of figures, proving statements like sum of any two odd numbers is always even where students need to identify an appropriate algebraic expression to represent the situation and manipulate it in order to reach the conclusion). These were done in the end so that students' understanding of symbolic algebra could be used as a tool in the tasks. We called this part 'reasoning *with* expressions'. Due to constraint of space, we do not discuss this part in the paper.

3 Description of the Research Study

The study was conducted as a design experiment (Cobb, Confrey, diSessa, Lehrer & Schuble, 2003) during the period 2003–2005 over five trials. The teaching–learning approach evolved through the five trials, with modifications made at the end of each trial based on students' understanding as revealed through the many tasks and our own understanding of the phenomena. The first two trials were part of a pilot study and are not discussed in detail in this report. The later three trials were part of the main study (henceforth, MST-I, MST-II and MST-III). The students for the main study trials came from two neighbouring schools (one English medium and one vernacular medium). These schools catered to children from low and medium socio-economic backgrounds. For MST-I, the students were randomly selected from a list of volunteers who had responded to our invitation to participate in the programme, and the same students were invited for MST-II and MST-III. All the three trials were held during the vacation periods of the school. MST-I was conducted immediately after their grade 5 examinations in April–May, 2004, MST-II when they were in the middle of grade 6 in October–November, 2004, and MST-III after the completion of grade 6 in April–May, 2005. Thirty-one students from the main study trials attended all the three trials—MST-I, MST-II and MST-III. Each trial consisted of 11–15 sessions of 1.5 h each. The students were taught in two groups, in the vernacular and the English language respectively by the research team members. The process of selection of the students, and the organization of the programme was broadly similar for the pilot trials, in which a different cohort of students participated. The topic area, the decision to use students' arithmetic knowledge to introduce beginning algebra and some of the structurally oriented tasks were developed during the pilot trials.

Both the schools from which students for the main study were drawn followed the syllabus and textbooks prescribed by the State Board. Of the topics covered by the study, the school textbook for grade 6 includes integer operations, evaluation and simplification of arithmetic and algebraic expressions in a traditional fashion — using precedence rules for arithmetic expressions (including expressions containing multiple brackets) and distributive property for algebraic expressions. Students also learn solution of simple linear equations, which was not part of our programme. Discussion with students and a review of their notebooks showed that only the vernacular medium school actually taught simplification of algebraic expressions in grade 6; the English school omitted it. The two chapters of arithmetic expressions and algebraic expressions take about 20–25 sessions of 35/40 min each in school. The classroom transaction is largely oriented towards stating of rules and demonstration of procedures followed by practice of some questions. Discussions about '=' sign or equality/equivalence of expressions and what we call 'reasoning with expressions' are not part of the school syllabus.

Data were collected through pre and post-tests in each trial, interviews after MST-II (14 students, 8 weeks after MST-II) and MST-III (17 students, 16 weeks after MST-III), video recording of the classes and interviews, teachers' log and coding of daily worksheets. The pre and post-tests consisted of around 25 items (both arithmetic and algebra) and took a couple of hours to complete. They tested for students' understanding of rules, procedures of evaluating/simplifying expressions, understanding of equality/equivalence of expressions and use of algebra to represent and justify/prove. The students were requested to show their working for the tasks. The students chosen for the interview after MST-II had scores in the tests which were below group average, average and above group average and contributed actively to the classroom discussions. The same students were also interviewed after MST-III along with a few additional students. Four of the students interviewed had each missed one of the three post-tests but had participated in all the trials. The interviews probed their understanding more deeply using tasks similar to those in the post test. The interview questions after MST-II were restricted to transformations of arithmetic expressions whereas after MST-III they included both arithmetic and algebraic expressions. Table 1 provides a summary of the tasks from the post test and the interviews which will be analysed in this paper.

The study aimed at achieving internal consistency in students' responses and reasoning and evolving a coherent and complete teaching approach for beginning algebra. We worked with multiple groups of students to increase the scope and the initial variability in the approach. We were interested in observing and analysing the effects of the refined teaching approach on students' understanding and reasoning in the context of arithmetic and algebra.

4 Evolution of the 'terms approach'

Some basic principles guided the teaching–learning approach. Students' understanding and intuitions/expectations in the context of arithmetic were used to guide their learning of symbolic expressions in algebra and their transformation. We framed concepts, rules and tasks which provided opportunities to work on their expectations, strengthening the right ones and correcting the incorrect ones. For example, students know intuitively that addition and multiplication are commutative but generalize it to subtraction and division; they also

Table 1 List of tasks given to the students in the tests and interviews

S. no.	Task	Example
1	Evaluation of arithmetic expressions	
	Simple expression (2 items in MST-I, 1 of each kind; 3 items each in MST-II and III, 1 with simple and product term, 2 with simple terms only)	$3+5\times6$ (simple and product terms) and $25-10+5$ (simple terms only)
	Complex expression (2 items in MST-I, 1 of each kind; 3 each in MST-II and III, 1 with simple terms, 2 with product terms)	$69-26-11+26-8$ (simple terms), $3\times16+16\times12-16\times7$ and $7\times18-6\times11+4\times18$ (product terms)
2	Simplification of algebraic expressions (2 items in MST-I, 1 in MST-II and 4 in MST-III)	$5\times x+16+7\times x-11$ and $x+15-13\times x-9$
3	Judging equality of expressions	
	Rearranging terms 'RT' (similar task for expressions with only simple terms and algebraic expressions; 3 items in MST-I, 1 of each type; 2 items in MST-III, 1 with simple and product terms and 1 algebraic expression)	Which of the following expressions are equal to the expression $23+17\times15+12$ (simple and product terms)?[a]—$17+23\times15+12$, $23+17\times12+15$ or $15\times17+23+12$, $23+12+17\times15$
	Other transformations 'OT' (similar task for expressions with simple terms only and algebraic expressions; 2 items in MST-II, 1 with simple and product terms and 1 algebraic expression; 2 items in MST-III, 1 with simple terms only and 1 with simple and product terms)	Which of the following expressions are equal to the expression $23-4\times6-9$ (simple and product terms)?—$23-(4\times6+9)$, $23-4\times6-8+1$ or $23-(7-3)\times6-9$, $22-4\times6-8$

[a] Item marked correct only when all expressions in the list were judged correctly

understand, without computation, that 34+29 is less than 34+31 but also think, using the same strategy that, 34–17 is more than 34–16. This was consistent with viewing arithmetic as a 'template' on which the understanding of algebraic symbols and operations on them could be built. Based on the literature review, we shifted the emphasis from mere computation of expressions to focusing on the structural aspect of expressions—stating the information contained in them, their value, the units which compose an expression and the transformations under which the value of the expression will remain the same. Students' understanding of algebra was developed by using and extending their experiences with symbols in arithmetic in specific ways. For example, the expression '4+3' did not only stand for '7' but also for the information that 'it is a number which is three more than four'. Multiple interpretations of '+' and '−' operations were similarly conveyed through meaning of sentences. Numbers were attached with the signs preceding them to denote signed numbers that could represent the amount of change—increase or decrease (e.g. change in the expression 26+13 vis-à-vis 25+14 can be represented by +1−1=0), or state a relation of 'greater' or 'less than' between two numbers/quantities. Further, the structural similarity between arithmetic and algebraic expressions was used to convey the need for similar rules of transformation. For example, the units in expressions like $2+3\times x$ and $2+3\times5$ or $2\times5+3\times5$ and $2\times x+3\times x$ are similar, thus warranting the same rules of transformation. In the first example, '3×5' needs to be treated like a unit and this constrains the possibility of the operation '2+3'. This constraint when learnt in the context of arithmetic expressions can be fruitfully used in the context of algebra, thus reducing the 'conjoining' error ($2+3\times x=5\times x$). In contrast to typical classroom environments which focus on correct procedures and answers, we ensured that reasoning underlined all tasks and activities, computational as well as non-computational.

The teaching approach aimed to strengthen students' understanding of procedures (knowledge of rules, conventions and procedures for working on expressions) and sense of structure (sense of the composition of the expression, how the components are related to the value of the expression and their relation among each other) in order to make a connection between the rules and properties used in arithmetic and the symbolic transformations in beginning algebra. The approach developed over multiple trials of the pilot and the main study. Only by the second trial of the main study (MST-II) did we succeed in evolving a formulation of rules and procedures in structural terms together with a set of tasks that could develop a strong understanding of procedures and structure and had the potential to connect arithmetic with beginning symbolic algebra. We first describe some of the difficulties in the earlier version of the teaching approach adopted in MST-I and then give a brief outline of the concepts and operations involved in the mature teaching approach.

In the MST-I, the approach emphasized the sequential operations in accordance with the precedence rules for evaluating arithmetic expressions—multiplication before addition, addition and subtraction to be operated sequentially, proceeding from left to right. To enhance students' understanding of structure of expressions, we introduced the concept of 'term' (units of expressions of which it is composed) in the context of evaluation of expressions to check if any precedence rule is required or one could compute sequentially from left to right. The concept was used largely for identifying and generating equal expressions; the transformation restricted to rearranging terms or numbers. It was expected that through the repeated use of 'terms' and precedence rules or sequential operations on arithmetic expressions, the students would be able to develop a sense of structure of expressions and would be able to abstract the constraints and possibilities of operations. However, we realized the use of 'terms' in both the contexts of evaluation and of identifying or generating equal expressions was rule governed and rigid; students made no connection between the two tasks. Moreover, the students failed to connect the precedence based/sequential procedures for evaluating arithmetic expressions and the rules for simplifying algebraic expressions. We defined 'like' and 'unlike' terms for simplifying algebraic expressions, which also required non-sequential, property-based transformation rules, like distributive property. This was hard for students to reconcile with their earlier instruction on evaluating arithmetic expressions and they continued to make structural errors and failed to constrain operations like $2+3\times x=5\times x$ ('conjoining' error) in the context of algebra. Evaluating and checking if two such algebraic expressions led to the same numerical value for a value of the letter was not of much help in convincing the students about the incorrectness of such a transformation. The students had not grasped the important idea that valid transformations ought to leave the value of the expression unchanged and thus transformed expressions will always be equal/equivalent. Thus, the concept of 'equality' seemed to be crucial for understanding transformations of algebraic expressions. It was factors such as this that prompted us to change to an approach that formulated rules and procedures in structural terms leading to a potentially closer integration of the responses to procedural and structural aspects.

In the more refined teaching approach used in MST-II, we used the concepts 'term', 'equality' and 'expression' and the operation 'combining terms' in order to establish the connection between procedure and structure on the one hand and between transformations on arithmetic and algebraic expressions on the other. We also made 'terms' of an expression visually salient by enclosing each term in a rectangular box. Terms are units of the expression demarcated by '+' or '–' signs; they can be transposed without changing the value of the expression and allow unambiguous parsing of expressions. For example, in the expression $12+3\times 5$, $+12$ and $+3\times 5$ are the two terms, with the former being called a simple term and the latter a product term. Terms are of two kinds: simple terms and complex terms. Complex terms include product term and bracket term (e.g. –(3+5)). The factors of the product term can be numerical (like $+2\times3$) or variable (like $+2\times a$). $+a$ or $-a$ can also be rewritten as product term ($\pm1\times a$) with one variable factor and the other factor being 1. Rules for evaluating and simplifying expressions were reformulated using the concept of 'term' and replaced the precedence rules for evaluation of expressions that were used in MST-I. Simple terms could be combined in a way similar to adding integers, using the compensation model: that equal and opposite terms cancel each other, positive term increases the value and negative term decreases the value of the expression (e.g. $+4-3=+1$). In expressions with simple and product terms with numerical factors, product terms needed to be simplified into a simple term before combining them with other simple terms (e.g. $+2+3\times5=+2+15=+17$). In expressions containing two or more product terms with a common factor, terms could also be combined by extracting the common factor (e.g. $+2\times5+2\times3=+2\times(5+3)=+2\times8=+16$). These rules provided flexibility in the order in which terms could be combined and also made it possible to discuss the possibilities and constraints of operations, thus deepening students' understanding of equality of expressions. This paved the way for identifying and generating equal expressions using a variety of transformations. It was expected that these rules would integrate transformations of arithmetic and algebraic expressions, as a result of enabling general understanding of procedures.

Even though we continued to test the students on each kind of task/item as described in Table 1 in each trial, we tried not to practice them in every trial. The expressions and the transformations were successively made more complex as the trials progressed, which is reflected in the test items. The new rules, which were only introduced in MST-II, were extensively discussed and used in the context of evaluating simple and complex arithmetic expressions with less time spent on simplification of algebraic expressions. During MST-III, we focused on developing understanding of evaluation of complex bracketed expressions (e.g. $23-4\times(2+3\times5)$) and simplification of algebraic expressions, together with tasks based on pattern generalization, proving and justifying. About six to seven sessions in each trial were devoted to tasks being discussed in this paper, and the remaining sessions focused on the various other tasks, such as understanding meaning of expressions, equality and integers; integer operations, evaluation of bracketed expressions and bracket opening rules, and using algebra as a tool for representing and in contexts of pattern generalization and justification.

5 Analysis of data and results

The performance of students in the post-tests of the three Main Study Trials in the various tasks together with the group average in each trial is given in Table 2. The table shows that students improved their performance over the trials in each of the tasks, leading to a better overall performance in these tasks, indicated by the group average.

Moreover, we can see each student's improvement across the trials in Fig. 1. Each column of dots indicates a single student's scores in the three trials. It shows that students' performance in MST-I is generally low even though the items in MST-I were fewer and simpler. The scores in MST-II are quite dispersed, some students scoring lower than MST-I and some higher than MST-I. The reasons will be clear from the more detailed discussion later on the nature of the items in the post-test of MST-II and students' performance on them. However, most of the students substantially improved their performance in MST-III

Table 2 Performance of students (as percentage of correct responses) in the different tasks in the post-tests of the three trials (N=31)

	Evaluation of arithmetic expressions		Simplification of algebraic expressions	Equality of arithmetic expressions		Equality of algebraic expressions		Group average
	Simple	Complex		RT	OT	RT	OT	
MST-I	68	50	29	58	–	61	–	4.7/9 (52%)
MST-II	89	54	19	–	53	–	62	12.7/19 (66.8%)
MST-III	85	74	84	77	74	87	–	15.7/20 (78.5%)

RT rearranging terms, *OT* Other transformations

and closed at a level higher than earlier. Also, most students in the lower half (lower right corner) during MST-I made significant improvements by the end of the three trials. A detailed analysis of each of the tasks reveals the aspects of students' understanding that improved over the trials. In the sections below, we describe the nature of changes that were seen in students' solutions and explanations.

5.1 Awareness of structure in evaluating arithmetic expressions

One of the important insights gained through the repeated trials was that seemingly procedural tasks such as evaluation of arithmetic expressions can be used to enhance students' awareness of structure as well as assess their structure sense. We discuss here responses of students to the tasks of evaluating arithmetic expressions and how these responses changed over time.

In the *pre-test* of MST-I, although many students correctly evaluated simple arithmetic expressions with 'simple terms' only (e.g. $19-3+6$, 64% correct), they did not follow the correct convention for precedence of multiplication while evaluating expressions with 'simple and product terms' (e.g. $7+3\times5$, 12% correct). In the post-test of MST-I and in the later trials, their performance was almost similar in both the expressions and there was a marked improvement in their performance with respect to expressions with both simple and product terms. The increase in performance was accompanied by a decrease in the structural errors such as computing the expression $7+3\times4$ as 10×4 ('LR' or left to right error) or $19-3+6$ as $19-9$ ('detachment' error, Linchevski & Livneh, 1999). Interviews with students also revealed students' abilities to use the structure of expressions to decide the validity of a transformation.

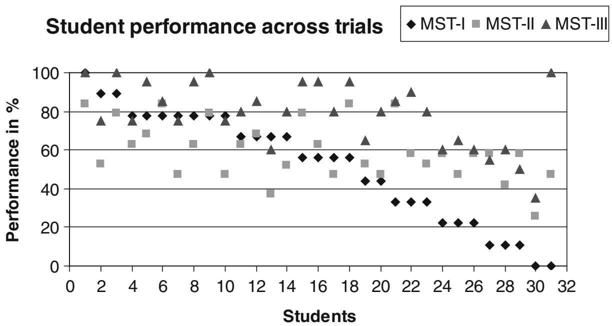

Fig. 1 Individual student's performance (as percentage of correct responses) across the three trials: MST-I, II and III (N=31)

They did not accept transformations like $5+3\times6=8\times6$ or $30+3$, $25-10+5=25-15$ by clearly indicating in the first case that factors of product terms cannot be separated and in the second case -15 is possible only if there is bracket around $(10+5)$.

The task of evaluating more complex expressions (e.g. $-28+49+8+20-49$ and $7\times18-6\times11+4\times18$) was created so as to de-emphasize precedence/sequential operations and encourage students to focus on the structure of the expressions, thereby using the relationships among the terms to make the computation simple. As seen in Table 2, students gradually improved their performance in evaluating the more complex arithmetic expressions. Table 3 shows the number of errors in the different items as well as the strategies used to solve the expressions across the trials. A strategy was categorized as 'relational (RS)' when students attended to the structure of the expressions and found efficient methods of combining terms so as to minimize calculations. On the other hand, a strategy was categorized as based on 'Precedence rules (PR)' when students followed a sequential order or gave precedence to a multiplication operation. The 'RS' strategies were spontaneously generated by students for expressions with only 'simple terms' while they had to learn the distributive property in order to apply relational strategies to compute expressions with only 'product terms' with a common factor. For the expressions with only product terms, we see a lower proportion of 'RS' strategy as compared to expressions with only simple terms.

With the evolution and adoption of the 'terms approach' in MST-II and MST-III, the students' work shows an increasing use of relational strategies and more flexible methods of evaluating expressions using relationships between the terms, thus showing an increased awareness of structure of expressions. In MST-I, the students did not take advantage of the relations which existed between the terms as seen in Fig. 2a, b. Rather, they proceeded to evaluate the expressions sequentially using precedence rules even after identifying the relationships between terms. The students' work in MST-II and MST-III (Fig. 2c–f) show that relationships were identified and terms were combined flexibly. (Fig. 2d shows an instance of what was the most common error in MST-II while combining product terms—students using the wrong sign when they had to combine product terms with a common factor.)

5.2 Simplification of algebraic expressions

We had reformulated the rules of transformation by highlighting the structure of expressions using the concept of 'term' in MST-II. We expected that this would lead to better connection between arithmetic and algebra and thus better performance in the algebraic simplification task. The data of Table 2 presented earlier show that students performed well on the task of

Table 3 Number of student responses by strategy in evaluating complex arithmetic expressions in the three trials (N=31)

Sample item	MST-I (post)		MST-II (post)		MST-III (post)	
	RS	PR	RS	PR	RS	PR
$69-26-11+26-8$	13 (4*)	11 (4*)	30 (4*)	1 (1*)	28 (3*)	3 (0*)
$3\times16+16\times12-16\times7$	–	–	21 (7*)	8 (3*)	21 (3*)	10 (3*)
$7\times18-6\times11+4\times18$	6 (1*)	15 (8*)	18 (17*)	9 (5*)	22 (7*)	9 (4*)

Arbitrary solutions and 'no attempts' not accounted for in the table

RS relational strategy, *PR* precedence rules

*Number of incorrect responses

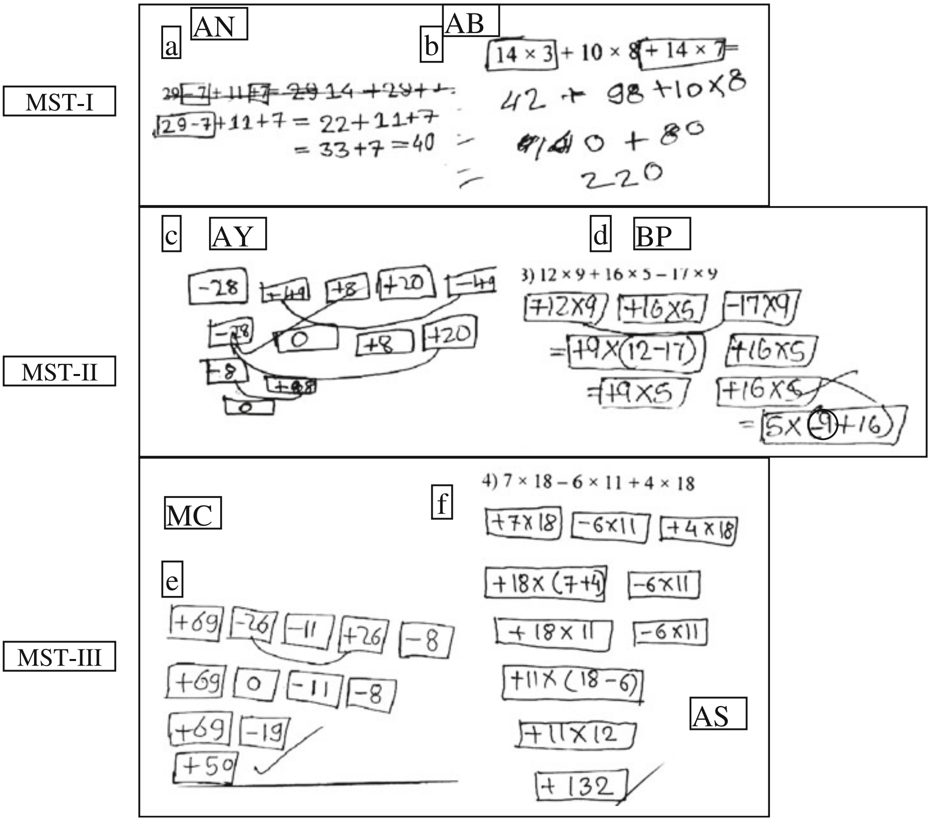

Fig. 2 Sample of students' evaluation of complex arithmetic expressions using 'relational' and 'precedence' strategies in the post-test of the three main study trials

simplifying algebraic expressions only in MST-III. In MST-I, students were still simplifying algebraic expressions using the traditional approach of identifying like terms (Fig. 3a, b). In MST-II, although students were able to simplify algebraic expressions using strategies similar to those used for arithmetic expressions, they made many errors; some due to the integer operations involved and some non-systematic errors like changing the term $x+15$ into $x\times15$ (Fig. 3d. About 16% of the students made this error). In MST-III (Fig. 3e, f), students were better at dealing with 'singleton' terms like 'x' and at integer operations, and most students could produce correct simplifications by extracting the common factors, similar to arithmetic expressions discussed earlier.

The increase in performance in simplifying algebraic expressions was accompanied by a decrease in structural errors. In MST-I, the conjoining error was repeatedly seen, throughout the steps of the simplification process (Fig. 3a). The conjoining error reduced from 31% in MST-I to 26% in MST-II and none in MST-III. Some other errors, such as change of sign, change of term, arbitrary solution process also reduced (18% in MST-I, 45% in MST-II and 10% in MST-III).

All students interviewed after MST-III could explain their procedures of simplifying algebraic expressions confidently. Students predominantly explained the simplification process by repeating the procedure they carried out, accompanied by statements like

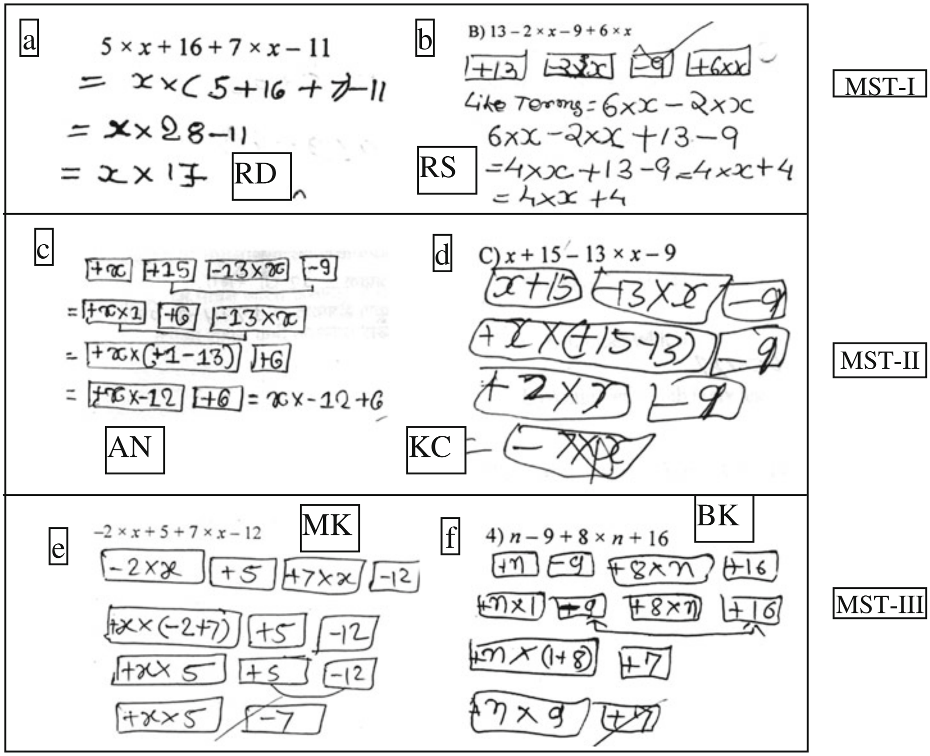

Fig. 3 Example of students' work on simplification of algebraic expressions in the post-test of the three trials

product terms can be combined by extracting a common factor and a simple and a product term cannot be combined. The student BK indicated the common factor 'a', and stated the fact that it will have the same numerical value for both occurrences, allowing the extraction of the common factor: 'it is $+5\times a+6-2\times a+9$. These two $[+5\times a-2\times a]$ are same therefore $+a\times(5-2)$'. When asked if $a\times 3+15$ can be simplified further, the student SV explained why this was not possible: 'Because $a\times 3$ is the product, you should not do $15+3$ and write. *The product term is to be done first*'. Both these students used their understanding of evaluating arithmetic expressions to transform algebraic expressions.

To check the robustness of students' understanding of simplification of algebraic expressions, they were further asked to predict the value of the original expression (e.g. $5\times a+6-2\times a+9$) given the value of the simplified expression (e.g. $a\times 3+15$) for a value of the letter (e.g. $a=4$). Eleven out of the 17 students interviewed, knew without resorting to calculation that the given expression and the final simplified expression are equal for all values of the letter. These students gave the following kinds of explanations:

SV: 'This expression $[5\times a+6-2\times a+9]$ and this expression $[a\times 3+15]$ are equal'

NW: 'This expression $[5\times a+6-2\times a+9]$ has been written in a simpler form'

BK: 'Because this is a product term and we do not know what the number *'a'* is. So we have to do it like this only'.

One student further articulated that just as the original expression could be simplified, similarly the simplified expression could be again converted into the original expression.

Six of the 17 students were not so confident and actually calculated the values of the original (by combining the product terms) and the simplified expressions for a given value of the letter, reaching the same conclusion as above. Except one, all these students appreciated the generality of the result after computation. Students' explanation for the equivalence of all the steps in the simplification procedure was largely drawn from their experience of evaluating similar arithmetic expressions and their abilities to mentally go through that procedure in the context of algebraic expressions.

5.3 Understanding of equality/equivalence of expressions

The task of identifying equal expressions from a list without computation was designed to directly tap into students' understanding of structure of expressions and encouraged them to use the idea of terms. The initial tasks were simple, with the only transformation applied to the expression being rearranging terms and numbers (MST-I). In MST-II and MST-III, the task was made more complex by including a variety of transformations. The pre-test data in MST-I showed that few students could identify equality of expressions without computation (15% correct). The percentage of correct responses in the post-test is similar over the first two trials, but the items in MST-II were more complex than MST-I (see Table 2 for performance data and Table 1 for types of transformations). The performance in identifying equality of expressions for both kinds of transformations (rearranging terms and others) improved in MST-III. (In MST-III, each item/expression involved only one type of transformation, while some items/expressions in MST-II combined different types of transformations.) Their performance in identifying equivalent algebraic expressions was slightly better than for arithmetic expressions, and slightly ahead of their ability to simplify algebraic expressions.

The performance varied depending on the transformation/s used. For example, only 25% of the students in MST-II could judge the equality of the expressions $18-27+4\times6-15$ and $8\times4-15+18-2\times4-27$. A large number of students could identify the equality of expressions like $18-27+4\times6-15$ and $4\times6-(27+15)+18$ (77% during MST-II and 93% in the post-test of MST-III).

The interviews after MST-II and III revealed that most students could infer the equality/equivalence of expressions by focusing on the structure of the expressions. A few students complemented this understanding with computations (computing parts of expressions) to be doubly sure. Students were given three to four alternative expressions to judge their equality with respect to a given expression. For example, the student AY in the interview after MST-II correctly judged the inequality of the expressions $49-37+23$ and $49-5-37-5+23$. Although a bit hesitant in the beginning, he gave a reason which incorporated both an understanding of structure of expressions and integer operations. He stated 'Because here -5–5 is extra. Had it been $-5+5$ then subtracting would have given us 0 but here it is both –5′.

In MST-III, a further probe was used when students correctly judged two arithmetic expressions as not equal—they were asked which expression was greater. Students used their understanding of procedures and structure sense to draw their conclusions. For example, JS concluded that the expression $24-13+18\times6$ was bigger than $24+18-13\times6$ because 'Here $+18\times 6$ is there which would give more answer, and here if we do -13×6 it will give less answer'. She ignored the marginal increase in value of $24+18$ compared with $24-13$ and attended to the significant difference caused due to the terms $+18\times6$ and -13×6.

Bracketed expressions presented some students with difficulties which arose largely as a result of their not seeing the equivalence between using the bracket as a precedence operation and bracket opening rules. This can be seen from AY's reason (in MST-III) for why he thought the expressions $24-13+18\times 6$ and $24-(13-18\times 6)$ to be 'equal' as well as 'not equal'. On the one hand he explained that 'Because three numbers are in the bracket, so the answer for these two [$13-18\times 6$] have to be found inside the bracket and whatever answer comes that has to be kept inside bracket and then do it with this [24] then you would get it not equal.'. On the other hand, he also held the belief that they were equal as 'if we open the bracket first then we get $+18\times 6$.'.

Students were not mechanically identifying equal expressions and connected the fact that equal expressions would have equal values. One student TJ remarked during the interview 'If they [expressions] were equal, then only [only then] their value would have been equal. And these expressions are different, the terms have been changed, therefore the answers will also be different'.

Students showed a good understanding of the fact that equivalent algebraic expressions are equal for all values of the letter. Ten students could straightway state this fact, whereas four others substituted the value of the letter to see if they were equal arithmetic expressions and three more calculated parts of the expression to conclude that the equivalent algebraic expressions will have equal values. Illustrating this with an example, BK stated that $13m-7-8\times 4+m$ and $13\times m-7-8\times m+4$ will not lead to the same value for $m=2$ as 'Because it is 8×4 [in the first expression], if it [the value of m] is 4 here then it would be the same value for both'. However, she agreed that the expressions $-7+4+13\times m-m\times 8$ and $13\times m-7-8\times m+4$ would be equal as 'Because, m is any number, if we put any number for that then they would be the same'.

Students used the same strategies and the same vocabulary to reason about transformations in arithmetic and algebra. Moreover, their understanding of procedures and structure complemented each other as seen in many of the students' responses.

6 Discussion and conclusions

The study proposes an approach to introduce symbolic algebra by supporting students' awareness of the structural similarity of arithmetic and algebraic expressions. The approach was motivated by the extensive literature indicating this to be an important idea for learning symbolic transformations in algebra. It supported and developed students' intuitive understanding of operations and numbers by introducing them to a structural way of perceiving expressions using a vocabulary of 'terms', different types of terms and equality/equivalence. Even though precedence rules are sufficient for evaluating expressions in arithmetic, they are not sufficient to transform algebraic expressions. One has to learn the constraints and possibilities on transformations in order to understand algebraic manipulations (Mason, Graham, Pimm & Gowar, 1985). Awareness of structure of expressions helps students understand these better, thus leading to a better understanding of rules and procedures. Arithmetic expressions provide the necessary background for this kind of learning, which can be generalized to understand transformations in algebra. In addition, in our programme, the visual salience of terms seems to have helped students both in computational and non-computational tasks.

The analysis of students' responses to a variety of tasks shows that students by the end of the third trial were able to (1) use their awareness of structure to evaluate arithmetic expressions in a flexible manner, reducing the structural errors, (2) use this awareness of structure consistently to judge equality of expressions, (3) understand that valid transformations lead to equal expressions and understand the conditions when the value of expressions remains the same, (4) use procedure sense and structure sense of expressions in a complementary manner to solve or reason about a task, and (5) use similar reasoning and transformations across arithmetic and algebraic expressions. In our view, these are major gains achieved towards the end of the programme. On the one hand, understanding about equality of expressions strengthened and introduced flexibility in the procedures of evaluating expressions. On the other hand, discussions regarding equality of expression, that is, transformations which could possibly keep the value of the expression the same, required the students to use procedural knowledge of evaluation of expressions. Students did not, however, compute the value of the expressions in order to judge their equality but compared the terms—a conceptual rather than an empirical approach. In these situations, they displayed a sense of reversibility and substitution—that an expression or a part of an expression can be replaced by a number and vice versa. Thus, their procedural and structural knowledge complemented each other and improved their overall understanding of expressions.

Given the vast literature which exists on students' understanding (or lack of it) of syntactic transformations (e.g. Kieran, 1992; Liebenberg, Linchevski, Sasman & Olivier, 1999; Liebenberg, Sasman & Olivier, 1999; Malara & Iaderosa, 1999; Cerulli & Mariotti, 2001) and some data from within this country (Banerjee, 2000), it is very unlikely that the traditional teaching–learning process in the school helps in developing such an understanding of expressions. Although it may be hard to separate the effects of more teaching from refined teaching in this study due to its design, it is worthwhile to attend to the improvements made by the students in the three trials. A part of the reason for the long duration of the programme was that the approach evolved and was modified in significant ways after MST-I, and in MST-II, time was spent on covering the same topics using the modified approach. Secondly, an approach that makes a smoother transition needs treating topics both in arithmetic and in beginning algebra in a more elaborate manner than is done in the traditional curriculum.

Another notable point that the study brought home is the inadequacy of the incomplete structural approach adopted in the earlier trials and the realization that the mere presence in the earlier trials (and in some earlier studies) of structural notions or tasks that focus on the structure of expressions, is not sufficient to bridge the gap between arithmetic and algebra. The earlier trials, which dichotomized procedural knowledge and structural understanding of expressions fell short of supporting students to make the transition to algebra. At that point, students could not see multiple ways of evaluating arithmetic expressions (by combining terms in different order) leading to the same value, they followed precedence rules and sequential evaluation procedures, ignoring relationships between the terms within an expression. They could not also generalize the knowledge of rules and procedures gained in the arithmetic context to algebra. What was needed was a focus away from computations towards understanding the structure of expressions, that is, identifying the components of expressions which contribute to its value and which remain invariant through valid transformations, thereby developing a deeper understanding about equality of expressions. Further, in the later trials rules of transformation formulated using structural notions were used to analyse both arithmetic and algebraic expressions, which allowed the students to connect arithmetic and algebra.

This study acted as a platform for us to observe the developing understanding of students with respect to syntactic transformations on expressions, given a certain kind of teaching–learning situation. The study was evolutionary in nature and therefore we can only point out the potential of the approach in creating meaning for symbolic transformation of expressions in both arithmetic and algebraic contexts, which students at this level can relate to. We do understand that algebra is not all about generalizing from arithmetic and is not only about symbol manipulation. It is yet to be seen how well these students use this understanding of symbolic algebra to deal with situations where algebra acts as a tool for proving, justifying, generalizing. Another direction in which the approach needs to be further developed is the incorporation of more complex symbol manipulation in the context of algebra.

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