FROM RELATIONAL REASONING TO GENERALISATION THROUGH TASKS ON NUMBER SENTENCES

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The current literature on algebra education calls for considering early algebra as a preparation for algebra teaching and learning. In this paper, we use tasks on number sentences as a context to explore the development of algebraic thinking in Grade 6 and 7 students. We discuss strategies used by students to solve these tasks and justifications or explanations given to support their responses. The findings of the study suggest that students move from purely computational strategies to relational reasoning and later generalised thinking. The use of box as a representation for number sentences supported students' thinking about structures and the movement from relational to generalised understanding. The study offers an instance of how early algebraic thinking in students in a classroom environment can be guided by students' thinking, conflict generation, and learning by consensual meanings.

Key words: algebra contexts, generalised thinking, number sentences, relational thinking, students' algebraic thinking

INTRODUCTION

Algebra is one of the most difficult topic areas in elementary school mathematics, with the use of letters for unknown numbers and variables presenting a major hurdle to students. The shift from working with numbers to working with letter symbols requires well designed instruction that facilitates the transition (Banerjee, 2008). There are are other identified challenges in learning of algebra such as understanding of equality, making generalisations, operating with letters, and flexibly dealing with procepts. Here we report a study which investigates students' thinking and learning of early algebra using number sentence tasks.

In a typical mathematics curriculum in India, children are first exposed to algebra in Grade 6. Algebra begins with discussion on the arithmetic properties (like closure, commutativity, associativity, and distributive property, identity) with the use of variables. The idea of variable is strengthened through pattern generalisation which leads to forming and solving simple linear equations. In Grade 7, solving algebraic equations becomes a major theme. Methods of solving linear equations (trial and error, balancing, and transposing) are followed by framing and solving equations from word problems. In Grade 8 students enter the world of quadratic equations and polynomials. The emphasis is on doing algebra or using algebraic notation. However, developing the tools for thinking algebraically is a perspective that does not yet find a place in the typical Indian mathematics curriculum. By algebraic thinking, we mean the act of deliberate generalisation and expression of generality (Lins & Kaput, 2004), analysing relationship between quantities, noticing structure, studying change, generalising, problem solving, modeling, justifying, proving, and predicting (Kieran, 2004).

Algebra research in 1980s and 1990s has focused on formulating stages for algebra learning and identifying student difficulties and its sources (Lins & Kaput, 2004). The later research conceptualises early algebra and explores teaching approaches to try it in classroom with younger children. *Early algebra* means building background contexts for problems to be solved using intuition or previous knowledge (Carraher & Schliemann, 2007), with the objective of exposing students to generalised mode of thinking while they are dealing with arithmetic. The development of relational understanding and focus on structures is central to early algebra. In the context of number sentences, relational (or structural) understanding means students attending to the *structure* of the sentence to decide what numbers make the number sentence true, not to carry out all the calculations indicated in order to determine the values of the missing number (Fuji & Stephens, 2008). Therefore, students who are able to use relational thinking to solve open number sentence problems consider the expressions on both sides of the 'equal to' sign while students with computational thinking view numbers on each side as representing separate calculations (Stephens, 2006 cited in Hunter, 2007). One of the ways in which development of structural thinking can afford processes of abstraction and generalisation (Mulligan, Vale, & Stephens, 2009) is exemplified in this paper.

In this paper, we share insights from our study, intended to explore number sentences (or "*expressions*") as a context to introduce early algebraic ideas to students with a focus on their progress to relational understanding and generalised thinking.

OBJECTIVES OF THE STUDY

The current study is a part of a larger study, which aims to support teachers' knowledge of students' thinking through the design of tasks that support teacher reflection. The phase of the study reported in this paper is the attempt to explore students' algebraic reasoning when exposed to *early algebraic* ideas through contexts like number sentences, pattern generalisation, proof and justification, etc. Based on students' reasoning, we plan to prepare student cases for discussion among mathematics teachers and teacher educators.

METHODOLOGY

The data collected was from a summer camp organised for Grade 6 and 7 students from three English medium schools in the vicinity of HBCSE. 68 students (37 boys and 31 girls) participated in the camp. The students were in the beginning of their academic year. The two groups of students were: 33 students (majorly Grade 6) in morning and 35 students (majorly Grade 7) in the evening batch. The summer camp continued for a period of 9 working days with a two-hour session every day. Two of the authors were the teachers for the camp. Data sources include classroom observations, teacher logs, and students' written and oral responses. The objectives of teaching were informed by literature on student difficulties and early algebra. Since there were different contexts used on different days of the summer camp, for the purpose of this paper, we elaborate on students' responses to tasks centered around number sentences. We viewed the videos of lessons on number sentences and identified episodes demonstrating the students' changing ways of dealing with number sentences. These episodes were transcribed for the purpose of reporting in the paper. Students' oral and written responses are analysed in the context of classroom discussion.

Task Design and Implementation

Teaching of algebra depends on how children are introduced to express qualitative relationships focusing on general mathematical relations (Fuji & Stephens, 2001). An important consideration for us in designing tasks for students was that the engagement in tasks should provide some evidence of children's capabilities of reasoning and abstract thinking. We knew that one of the useful routes is working on algebraic expressions through the broadening of arithmetic ideas, which can create more opportunities for student learning (Banerjee, 2008). Since it is the first time that these students are exposed to algebraic thinking (or algebra), we were keen on using number sentences as a beginning context. We were curious to find out the affordances of the number sentences task as students' reasoning progressed. We also used tasks of 'comparing quantities to elicit multiple strategies from students' (Naik, Banerjee & Subramaniam, 2005).

The beginning tasks on number sentences (Table 1) were designed to understand students' identification of the relations among numbers and make their thinking explicit. As the tasks progressed we observed students' movement from procedural ways to reasoning structurally while solving number sentences, i.e. reasoning based on the relations between the terms in the numerical expressions. This observation guided us to design later tasks in order to give students an opportunity to move from relational thinking to generalised thinking.

Objective	Beginning Tasks on Number Sentences	Later Tasks on Number Sentences
Making students to explicate/ verbalise their (relational) thinking	76 + 47 = ___ + 48 876 + 547 = ___ + 878	876 + 547 = $\square$ + 878 a + b = a – 1 + ___ a + b = a + $\square$ + b – $\square$
	57 – 41 = 56 – ___ 457 – 341 = ____ – 342	457 – 341 = 456 – $\square$ a – b = a – 1 – ___ a – b = a – $\square$ – b + $\square$

Table 1: Tasks used in the four Teaching Sessions

The beginning tasks for completing number sentences were guided by the notion of equality and relations in numbers. We started with examples like 76 + 47 = ___ + 48 and soon shifted to using larger numbers in order to direct students' attention to the structure of number sentences. The initial responses of students to these tasks were largely computational. A majority of the students added the two numbers on the same side of equal to and subtracted the number on the other side from the sum. As the students started identifying and talking about relations in numbers on either side of equal to, they were introduced to the need for expressing any number in the form of relations they identified. The notation of *box* emerged as a placeholder for an unknown number in this process. The reason for using a *box* instead of a letter as an unknown was indications from the research literature that variables are difficult for students to decipher as numbers. The *box* was introduced as a place-holder representing 'a place for any number', or precisely as students said it '*any number can go inside it*'. We found that the box representation gave freedom to students to talk about generalisations and facilitated mathematically rich discussions around the given equations. Apart from filling the missing value in addition and subtraction number sentences, we also had tasks on true/ false sentences and creating sentences individually and in groups.

DATA ANALYSIS AND FINDINGS

Before presenting the transition in students' thinking from computational to relational to generalised thinking, we would like to describe the classroom culture and pedagogic moves which supported us in knowing about students' thinking and therefore take decisions while teaching in classroom.

Typically, each teaching session began by asking students to respond to a set of problems either in a worksheet or on the chalk board. Students could choose to work either individually, with partners or in groups. After they finished spending some time on the problem, they would explain their method to the whole class, during which other students and teacher posed questions if they were not convinced. After one strategy had been discussed and agreed upon, students who proposed a different strategy came up and explained their strategy. The blackboard was used to record different strategies proposed by students. There was a discussion on the effective strategy and what makes some strategies more effective than others. We noticed the evolution of a classroom culture where students would refer to each others' strategies by citing their names, pose questions when in doubt, or comment on each others' strategy.

Students were introduced to the idea of number sentences and were encouraged to explicate the reasons for the truth of a number sentence. Students' explanations served as a way for teacher(s) to know about their prior knowledge and the connections they make, their approach to problem solving, etc. There were also discussions on the significance of (thinking and) asking why to find the reasons for responses. The accepted 'reasoning' was consensually defined as trying to explicate what we are thinking when we solve a problem and why we think the strategy we choose works.

From Computational to Relational (or Structural) Thinking

In the beginning, almost all students had a computational approach towards addition number sentences. Students carried out the calculations for pair of numbers on one side of the “equal to” sign and taking away the number on the other side to find the number in the blank (Table 2).

S23[1] (using computations)	S49 (blank as variable)	S18 (using variable x)
53 + 38 = 54 + __	**48 + 39 = 40 + ___**	**53 + 38 = 54 + __**
53 91	87 = 40 + ___	53 + 38 = 54 + x
+<u>38</u> - <u>54</u>	87 – 40 = ___	53 + 38 – 54 = x
91 <u>37</u>	47 = ___	91 – 54 = x
		<u>37</u> = x

Table 2: Student Responses to Number Sentences (Session 1)

While doing this procedure, all the students were convinced with the rule that '*sign changes when we move from one to the other side of equal to*'. The conversation that follows was carried out with several students in interviews as well as during classroom teaching.

Classroom Excerpt 1: Procedural Understanding

> Number sentence: **48 + 39 = 40 + ___**
> Student: 48 plus 39 is 87. 40 is subtracted from 87.
> Teacher: Okay, how?

1 Students have been named from S1 to S68. Henceforth, this naming is used to refer to individual students.

Student: When it goes to the other side it will be 87 minus 40. So answer is 47.
Teacher: How does the plus becomes minus when it goes to the other side?
Students: It is a rule.
Teacher: But why does it work?
Students: It is a rule only. It is true.
Teacher (to class): Are you all convinced about it?
Students (in chorus): Yes

The proposition of sign change was treated as a given rule. Neither did the students raise a question on why this is true nor did they know the reason. It was difficult to make them think about the need to know why this is true and holds for any equation.

Another approach that exemplified the use of procedures was replacing the blank with a specific letter. Many Grade 7 students substituted the blank with an *x* stating '*let the blank be x'* followed by which they solved the equation to find the value of *x*. They continued to think that any unknown should be replaced by the letter *x* and then all the numbers should be taken on the other side of *x* and computed (S18, Table 2). Students using this procedure had the same idea about sign change as others, using the rule without knowing the reason.

Also, while going through students' work, we found that majority of students did not face the commonly reported difficulties in literature like interpreting 'equal to' as 'something to do signal' or as 'closure of expression'. We think that not making such errors might have been due to the students being older and instruction that they have had.

The computations done by students assured that students got the correct answer but as Fuji & Stephens (2001) suggest, the goal is to focus children's attention on the underlying mathematical structure exemplified by that sentence. Students figured out the uniqueness of number sentences being posed to them in the next session. Classroom Excerpt 2 marked the beginning of looking at relations between numbers in a given set of number sentences.

Classroom Excerpt 2: Towards Relational Understanding

Teacher: There is something similar in all the number sentences, right?
Students: Yes.
Teacher: There is something common, what is it?
Different answers from students:
Students: All of them have plus, dash (blank), some numbers, same way to reach answer
S7: Teacher, in each sum of the three numbers...two numbers are very close

The students were convinced of the similarity stated by S7 and as the discussion went on, another student S2 expressed that 'a*ctually equal to is like a balance. If we take away something from one side we have to give it back. So we take it away from the other side also or add it to the same side*'. This was a crucial juncture and students readily accepted this idea. Despite this discussion, we found many students still using computations to solve number sentences. On probing, we discovered that students felt that computation was a secure way to get a correct answer. However, the new discourse in the classroom was about effective strategies, relation between numbers, equal to as a balance, etc. It was interesting to note that the students using procedural approaches realised that the efficient strategy was to compare numbers on either side of equal to and so their justifications changed in the later sessions. We found that in the second session, a number of students started using both the methods to solve a number problem, where they treated one way to solve and the other to verify their answer.

S36 (relational then procedural)	S4 (procedural then relational)

79 + 46 = ___ + 48
48 – 46 = 2   79 + 46 = 125
79 – <u>77</u> = 2   125 – 48 = <u>77</u>
	62 + 19 = ___ + 20
	  62 81 20 – 1 = 19
	+<u>19</u> - <u>20</u> or 62 – 1 = <u>61</u>
	  81 <u>61</u>

Table 3: Students' Use of Relations and Computations (Session 2)

The evidence from comparing students' responses from Session 1 and 2 showed that almost all the students used computations to solve number sentences in the first session. But as we moved, we witnessed a change in students' strategies and reasoning from procedural to beginning relational thinking.

Nature of Relational Thinking in Students' Reasoning

Students continued to use their methods (computational and/ or relational) for fill in the blank problems, true/false number sentences, and for creating and solving their own sentences. But we found students using different representations to express relations in numbers. There were explanations with words, using diagrams, using numbers and computations, writing more than one reason, etc. (Table 4). Students stated that these solutions (using relational thinking) made their responses quicker and we found that they were gaining confidence in the use of relations. Often they would also look for similarities in different representations to justify their strategy.

S31	S33	S64
45 + 22 = <u>23</u> + 44 +1 -1	**27 + 32 = ___ + 28** 27 + 1 = 28 32 + 1 = 33 Therefore, ___ = 33	**62 + 19 = __ + 20** "Answer is 61. If we subtract a number from one of the numbers and add the same number to the other, answer will be same."

Table 4: Responses on Number sentences (Session 3)

The idea of 'equal to' as a balance was also getting strengthened. There were other related ideas which were emerging. Some students started using the diagrammatic representation of the balance between the two numbers to show commutative property.

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264 + 191 = 191 + 264   (presented to class by S 49)

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Students started using this explanation to support other claims for instance: S40 wrote that *'20 = 20 because equal to is a balance and on each side equal weight should be there'*. Also, the discussion on the sign change was revised and students now could make sense of the changing sign with the explanation of balancing. The justification for sign change was extended from number sentences (with the relation between two numbers) to any two expressions on either side of equal to (S64, Table 4).

From Relational to Generalised Thinking

After students attained a level of comfort in working with number sentences, the trajectory took a different turn. A student in the beginning of the fourth session said that '*this (pattern) works for all the numbers... I take any number add one to the first number and subtract one from the second number*, *I get the same answer*'. At this point, there was a discussion on whether it is possible to express this relation as a generalised mathematical statement. This was accompanied with the introduction of a new representation called *box*. The different levels in which this generalisation happened in class was

Level 1: a + b = a + 1 + b – 1

Level 2: a + b = a + 5 + b – 5, (5, 6 or 10)

Level 3: a + b = a + 100 + b – 100

Level 4: a + b = a + $\square$ + b – $\square$ = a – $\square$ + b + $\square$

The sequence of number sentences made students generalise with the *box* as representing any number. When asked about the conditions under which the above number sentence will be true, students became more specific. The conditions stated by them were '*a and b hold the same value on either side of equal to and the box refers to the same number in a number sentence'*. This was extended to saying that '*the sign of the numbers inside box should also be the same and it can be a fraction, decimal or integer'. The box thus signified the representation for any number.* They extended their understanding of a and b as any whole numbers to them belonging to different classes of numbers. It was interesting to see the enthusiasm with which students pursued the idea of generalisation with *box* as a generalised number and proving that 'the sum of the two numbers remains the same if any number or box is added to the first number and the same is subtracted from the second number' (Level 4).

Thus, it was found that the strategies used by students while justifying number sentences involved complex interweaving of computational-structural understanding, articulation of relational thinking and the movement to generalisation. We saw shifts in students' reasoning from computational thinking to developing relational understanding to the need for generalised statements and their proofs. The later sessions focused on the ideas of justification and proof of generalised statements.

CONCLUSIONS

Achieving generalisation is a cornerstone in learning algebra at the school level. The analysis of students' work on number sentences and the trajectory in their reasoning verified the potential of these tasks as a context to trigger relational reasoning. The trajectory of working on number sentences was seen as starting from procedural (or computational) to relational to generalised thinking, supporting the view that understanding of structures is a key to generalisation (Mulligan et al., 2009). Generalised reasoning reinforced students' idea of equations as a *balance* where they were found demonstrating compensation of quantities symbolically. Along with the role that classroom culture and students' prior knowledge played in development of this trajectory, we also identified how intermediate resources such as use of a box supported the trajectory towards generalised thinking. The use of box provided liberty to students to put anything inside it - small, big or even a negative number. We think that box represents a partial symbolisation of the concept of variable and students found it easier to relate its use both as an unknown and a variable.

Number sentences is a powerful context and can be integrated with the existing Indian curriculum. Students' use of number sentences brings forth the algebraic nature of such arithmetic tasks. However, the movement from computational to generalised thinking in a flexible mode entails a significant role of the teacher, including identifying the appropriate prompts, and planning for the unexpected student responses. Understanding the teacher's role in the trajectory of students' thinking is one of the crucial components of teaching algebra in school.

Acknowledgements

We extend our sincere thanks to all Mathematics Group Members at HBCSE for their support in different capacities, in sharing their insights and helping with the logistics of the camp.

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